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Asymptotic and transient behaviour of nonlinear control systems

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Summary

In this thesis, the problem of controlling both transient and asymptotic behaviour of solutions of functional differential equations is addressed. The work begins, in Chapter 1, with an introduction to basic control theory principles that will be used throughout. This is followed by the introduction of a class of nonlinear operators in Chapter 2 and the development of suitable existence theories for the associated system classes of functional differential equations and inclusions in Chapter 3. A discussion is provided, in Chapter 2, describing diverse phenomena, such as delays and hysteresis, that can be incorporated in the class of operators.

Chapters 4–7 cover four areas of research. Chapter 4 examines the asymptotic and transient behaviour of nonlinearly-perturbed linear systems of known relative degree; a continuous feedback strategy is adopted and an approximate tracking result is presented. In Chapter 5 the class of systems considered is expanded to a large class of nonlinear systems and a continuous feedback strategy is implemented in order to achieve approximate tracking.

In Chapters 6 and 7 attention is restricted to systems of relative degree one, but this limitation is compensated for by targeting an exact asymptotic tracking result. The first investigation, in Chapter 6, involves a potentially discontinuous feedback controller applied to a class of nonlinear systems, with comparisons made to an internal model approach. Asymptotic tracking and approximate tracking are developed in unison within a framework of functional differential inclusions. Finally, in Chapter 7, a continuous controller is applied to single-input, single-output, nonlinear systems with input hysteresis.

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Notation

$\mathbb{N}$	the natural numbers,
$\mathbb{R}$	the real numbers,
$\mathbb{R}_+$	$:= [0, \infty)$,
$\mathbb{C}_+$ ($\mathbb{C}_-$)	the open right (left) half complex plane,
$\mathbb{R}[s]^{n \times n}$	the set of $n \times n$ matrices over the ring of real polynomials,
$\mathbb{R}(s)^{n \times n}$	the set of $n \times n$ matrices over the field of rational real functions,
$\lvert x \rvert$	the absolute value of a real number (modulus of a complex number),
$\langle \cdot, \cdot \rangle$	the Euclidean inner product,
$\lVert x \rVert$	$:= \sqrt{\langle x, x \rangle}$, the Euclidean norm for $x \in \mathbb{R}^n$,
$\mathbb{B}_\delta(x)$	$:= \{ y \in \mathbb{R}^n \mid \lVert x - y \rVert < \delta \}$, open ball, radius $\delta > 0$, center $x \in \mathbb{R}^n$,
$\mathbb{B}_\delta$	$:= \mathbb{B}_\delta(0)$,
$\overline{A}$	denotes the closure of $A \subset \mathbb{R}^n$,
B^T	denotes the transpose of the matrix $B \in \mathbb{R}^{n \times m}$,
$GL_n(\mathbb{R})$	denotes the set of invertible, real-valued, $n \times n$ matrices,
$\mathrm{spec}(B)$	the spectrum (set of eigenvalues) of $B \in \mathbb{R}^{n \times n}$.

For an interval $I \subset \mathbb{R}$, we denote the following spaces of functions $x \colon I \to \mathbb{R}^q$ by:

$C(I, \mathbb{R}^q)$	continuous functions,
$AC(I, \mathbb{R}^q)$	absolutely continuous functions,
$AC_{\mathrm{loc}}(I, \mathbb{R}^q)$	locally absolutely continuous functions,
$C_{\mathrm{pm}}(I, \mathbb{R}^q)$	continuous, piecewise monotone functions,
$L^1(I, \mathbb{R}^q)$	integrable functions $x \colon I \to \mathbb{R}^q$ with norm
$\lVert x \rVert_1$	$:= \int_I \lVert x(t) \rVert dt < \infty$,
$L^1_{\mathrm{loc}}(I, \mathbb{R}^q)$	measurable, locally integrable functions,
$L^\infty(I, \mathbb{R}^q)$	measurable, essentially bounded functions with norm
$\lVert x \rVert_\infty$	$:= \text{ess-sup}_{t \in I} \lVert x(t) \rVert$,
$L^\infty_{\mathrm{loc}}(I, \mathbb{R}^q)$	space of measurable, locally essentially bounded functions,
$W^{1,\infty}(I, \mathbb{R}^q)$	the Sobolev space of bounded functions $x \in AC(I, \mathbb{R}^q)$ with derivative $\dot{x} \in L^\infty(I, \mathbb{R}^q)$ and norm
$\lVert x \rVert_{1,\infty}$	$:= \lVert x \rVert_\infty + \lVert \dot{x} \rVert_\infty$,
$x\vert_J$	denotes the restriction of $x \colon I \to \mathbb{R}^q$ to $J \subset I$.

Space of functions used in this thesis:

Let $\mathcal{G}$ be a domain in $\mathbb{R}_+ \times \mathbb{R}^l$, that is, a non-empty, connected, relatively open subset of $\mathbb{R}_+ \times \mathbb{R}^l$. A function $f\colon \mathcal{G} \to \mathbb{R}^q$ is said to be a Carathéodory function, if the following conditions are satisfied:

- $f(t, \cdot)$ is continuous for all t,
- $f(\cdot, y)$ is measurable for each fixed y,
- for each compact set $\Lambda \subset \mathcal{G}$, there exists $\kappa \in L^1_{\text{loc}}(I, \mathbb{R}_+)$ such that

$$\|f(t, y)\| \le \kappa(t) \quad \forall (t, y) \in \Lambda.$$

The following function spaces will also be used:

K	$:=\{\gamma \in C(\mathbb{R}_+, \mathbb{R}_+) \mid \gamma(0) = 0$ and γ is strictly increasing$\}$,
K_∞	$:=\{\gamma \in K \mid \gamma(s) \to \infty$ as $s \to \infty\}$,
KL	functions $\theta\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that, for each fixed $s \in \mathbb{R}_+$, the map $\theta(\cdot, s) \in K$ and the map $\theta(s, \cdot)$ is decreasing to zero.

Chapter 1

Introduction

1.1 Control theory

Control theory is the area of applied mathematics concerned with analyzing and synthesizing the behaviour of dynamical systems with inputs. By constructing appropriate inputs, referred to as controls, the aim is to influence the behaviour of a system in some desirable manner. Frequently, such controls are used to force the output of a system, or class of systems, to perform a specific objective such as tracking a reference signal.

There are many facets to the study of control theory. One area, with strong links to the calculus of variations, involves optimizing the behaviour of a system for which a good mathematical model is known. A second branch, and focus of this thesis, involves the study of uncertain systems of a known class where the idea is to construct a single control strategy capable of achieving the desired objectives for every member of the class. Such controls are known as *universal controls.*

In this thesis, universal control strategies are designed for a variety of system classes with two specific types of control objective in mind. Firstly, we seek to control the asymptotic behaviour of solutions, meaning that the long-term performance of solutions achieves some prescribed goal. The second type of control objective deals with the transient behaviour of solutions. The two objectives, paired together, ensure that solutions not only attain a long-term goal, but perform in a prescribed manner throughout. Both performance aspects are expanded upon in Section 1.2.

1.1.1 Closed-loop feedback

An essential concept in the development of universal control strategies is the idea of feedback. A control scheme is sought in which, in the words of Fleming [16, Chapter 2],

1.1.3 Motivating example

As an example, consider a simple pendulum with input force u as illustrated in Figure 1-2. In this basic example, the effects of friction and air resistance are ignored and it

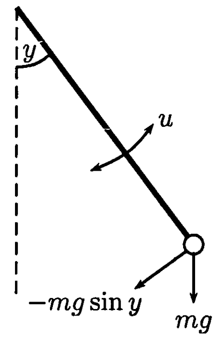


Figure 1-2: A simple pendulum.

is assumed that the mass is concentrated at the end point of the pendulum. Setting y to be the angle of rotation, measured anticlockwise, the governing equation for the pendulum is given by the following nonlinear differential equation

$$m\ddot{y}(t) + mg\sin y(t) = u(t), \qquad y(0) = y^0, \tag{1.1}$$

where g is the gravitational constant and the mass of the pendulum is denoted m. An anticlockwise force exerted by the control is considered to be the positive direction.

By measuring the angle of rotation, the output value of the system (y) can be compared to a desired command signal and an input constructed using a suitable feedback controller. Therefore, a closed-loop feedback problem can be considered in which the aim is to control the behaviour of the pendulum. This example is not, itself, of great importance in this thesis, however the particular structure of the formulated system will motivate the discussion of several key structural assumptions later in this Chapter.

A variety of control objectives can be formulated and, in each case, we seek a control strategy capable of forcing the output to achieve a particular task. In the example above, a typical asymptotic control objective might be to stabilize the output y to a chosen fixed value, such as the angles $y = 0$ or $y = \pi$, at which the pendulum is exactly vertical (equilibrium states when no input is applied). In addition, an objective relating to transient behaviour could involve the development of a control that not only ensures that the pendulum reaches a specific state but also maintains the pendulum within a set distance from that state throughout the evolution of the system output. In Section 1.2, a discussion is provided detailing the main control objectives considered in this thesis.

1.2 Control objectives

Two main types of control objective are considered. Primarily, controllers are designed to influence the asymptotic performance of solutions to a variety of systems. The second objective is to achieve prescribed transient behaviour of the solutions. The following sections discuss the various aspects of each control objective.

1.2.1 Asymptotic performance

A great deal of attention has been paid to the asymptotic behaviour of solutions to various control systems. An asymptotic objective with considerable history is the problem of output stabilization. In this case, a controller is designed to ensure that, as $t \to \infty$, the system output $y(t) \to 0$. A wide range of papers tackle the problem, with an early contribution to non-identifier based adaptive control (by which we mean control strategies that involve no attempt at system identification) appearing in 1978 through the efforts of Feuer and Morse [14].

1.2.2 Exact asymptotic tracking

A natural evolution from the stabilization objective is the problem of tracking a reference signal. Here, the output is required to track asymptotically any chosen reference function (denoted r throughout this thesis) from a class of signals $\mathcal{R}$, in the sense that, as $t \to \infty$, $y(t) - r(t) \to 0$. We refer to such performance as exact asymptotic tracking. By writing the objective in this form, it is clear that, via the coordinate change $e(t) = y(t) - r(t)$, the problem of asymptotically tracking a reference signal can be reduced to stabilization of the error signal e.

Two control methods frequently used to achieve exact asymptotic tracking objectives are as follows: (i) continuous output feedback wherein an internal model, capable of reproducing the class of reference signals, can be incorporated into the feedback loop, (ii) discontinuous output feedback without recourse to an internal model. The two methods are given further introduction in Sections 1.7.1 and 1.7.2 respectively. A (potentially) discontinuous feedback strategy will be used in Chapter 6 and comparisons will be made with the internal model approach.

1.2.3 Approximate tracking

In some situations, particularly those in which practical issues are considered, an exact asymptotic tracking objective may not be realistic. Instead, the idea of *approximate tracking* can be investigated. For some arbitrary $\lambda > 0$, an output feedback strategy is sought which ensures that, for every reference signal $r \in \mathcal{R}$, the tracking error $e(t) = y(t) - r(t)$ is ultimately bounded by λ (that is, $\|e(t)\| \leq \lambda$ for all t sufficiently large, often referred to as λ-tracking). The results relating to systems with known relative degree (in Chapters 4 and 5) involve an approximate tracking objective.

1.2.4 Transient behaviour

The trajectories of an asymptotically stable linear system may deviate significantly from the origin, since guaranteed long-term performance (convergence to zero of all solutions) does not exclude the possibility of large excursions in state space. In the tracking case, this equates to a deviation by the output from the reference signal to be tracked and hence the transient error may take large values. In both the stabilization and tracking cases, such behaviour, particularly in practical situations, may be highly undesirable and so, in this thesis, attention is paid to the transient behaviour of solutions to differential equations.

In [20, Section 5.5], transient behaviour of linear systems taking the form $\dot{x} = Ax$ is discussed in detail and a new stability concept, referred to as (M, β)-stability is introduced, combining information about decay rate and transient behaviour. Transient amplification of initial state perturbations is quantified through the notion of a transient bound and the relationship between the transient bound and decay rate is discussed.

Papers examining the transient behaviour of solutions to systems of equations are less common in the literature than those pertaining to tracking and stabilization objectives. However, we highlight two papers [48] and [26], discussed in more detail later in this chapter, in which controllers capable of shaping the transient behaviour of solutions are implemented. Some discussion of the transient response of control systems is also provided in [3, Section 4.3] and illustrated using a speed control example. Other papers considering the improvement of transient performance in tracking control include [38], for example.

1.3 The performance funnel

In order to control both the transient and asymptotic behaviour of solutions we introduce the concept of a performance funnel, see Figure 1-3. The performance funnel was first utilized in [26] and a full description is provided in the following definition.

Definition 1.3.1 (Performance Funnel) *The performance funnel is given by*

$$\mathcal{F}_\varphi := \left\{(t,e) \in \mathbb{R}_+ \times \mathbb{R}^m \middle|\ \varphi(t)\,\|e\| < 1\right\}$$

associated with a function φ (the reciprocal of which determines the funnel boundary), belonging to one of the following spaces of functions:

$$\Phi := \left\{\varphi \in W^{1,\infty}(\mathbb{R}_+,\mathbb{R}) \,\middle|\, \varphi(0)=0,\ \varphi(s)>0\ \forall\ s>0,\ \liminf_{s\to\infty}\varphi(s)>0\right\}\ ,\ \mathit{or}$$

$$\Phi_\lambda := \big\{\varphi \in AC_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}_+)\ \big|\ \varphi(0)=0,\ \ \varphi(s)>0\ \forall s>0,\ \ \liminf_{s\to\infty}\varphi(s)=1/\lambda,$$
$$\exists\ c>0:\ \ \dot\varphi(s)\le c[1+\varphi(s)]\ \mathit{for\ a.a.}\ s>0\big\}\,,$$

with the convention that, if $\lambda = 0$, then $1/\lambda := \infty$ (and so $\varphi(t)\to\infty$ as $t\to\infty$).

The aim is an output feedback strategy ensuring that, for every reference signal $r \in \mathcal{R}$, the tracking error $e = y - r$ evolves within the funnel $\mathcal{F}_\varphi$.

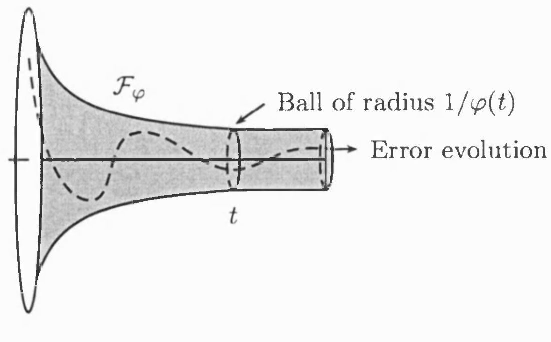


Figure 1-3: Prescribed performance funnel $\mathcal{F}_\varphi$.

Remark 1.3.2

(i) The assumption that $\varphi(0) = 0$ ensures that the tracking error can take any value at the initial point (there is no requirement that a bound exist on the initial data). The conditions $\varphi(s) > 0$ for all $s > 0$ and $\liminf_{s\to\infty} \varphi(s) > 0$ make sense since they eliminate the possibility that the funnel has, or expands to, an infinite radius ($\varphi(s) = 0$) for non-zero s.

(ii) In Chapters 4 and 5 an approximate tracking objective is sought, in which case we choose $\varphi \in \Phi$. The key assumption to note is that, since $\varphi \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$, the function φ is bounded and so the function $1/\varphi$, used to describe the funnel boundary, will be bounded away from zero.

(iii) In Chapters 6 and 7, the case of unbounded φ is accommodated. By choosing $\varphi \in \Phi_\lambda$, the aim is to include the possibility of an exact asymptotic tracking objective, occurring in the case $\lambda = 0$. Therefore $\varphi \in AC_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}_+)$ is sufficient, but an additional restriction on the derivative of φ is imposed. In this situation, by remaining within the performance funnel, the error must decay to zero.

(iv) Observe that φ is not required to be monotone, see Figure 1-4.

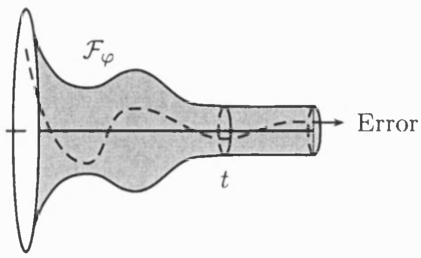


Figure 1-4: A variation on the performance funnel $\mathcal{F}_\varphi$.

Example 1.3.3

(i) Let φ satisfy $\liminf_{t\to\infty} \varphi(t) > 1/\lambda$, then evolution within the funnel ensures that the tracking error e is ultimately bounded by λ (that is, $\|e(t)\| \leq \lambda$ for all t sufficiently large).

(ii) If $\varphi \in \Phi$ is chosen as the function $t \mapsto \min\{t/T, 1\}/\lambda$, then evolution within the funnel ensures that the prescribed tracking accuracy $\lambda > 0$ is achieved within the prescribed time $T > 0$.

(iii) Suppose $\varphi \in \Phi_\lambda$ and $\varphi(t) \to \infty$ as $t \to \infty$; the example $\varphi\colon t \mapsto t^p$, $p \in \mathbb{N}$ suffices. Evolution through the performance funnel then implies that an exact asymptotic tracking objective is achieved and so $e(t) \to 0$ as $t \to \infty$.

1.4 Prototype class of systems

When developing a universal control, an aim is to consider as large a system class as possible and so, in this section, beginning with a simple prototype class of systems, we build towards the larger system classes considered in this thesis.

1.4.1 Single-input, single-output systems

One of the most basic classes of control systems is that of linear, single-input (denoted u), single-output (denoted y) scalar systems of the form

$$\dot{x}(t) = ax(t) + bu(t), \qquad x(0) = x^0 \in \mathbb{R}, \tag{1.2}$$

$$y(t) = cx(t), \tag{1.3}$$

where $a, b, c \in \mathbb{R}$. It is natural to assume that $cb \neq 0$. A variety of control objectives and strategies have been investigated for this class of systems. Utilizing output feedback, in which the input u will be a function of the output y, the most common objective is to ensure that $x(t) \to 0$ as $t \to \infty$. Pioneering work by Morse [49] and more recent research by Helmke and Pratzel-Wolters in [19] provide a detailed treatment of such systems.

In the case wherein the values of a, b and c are known, the output feedback strategy $u(t) := -ky(t)$ can be implemented, exploiting a *high-gain* property of the system class: if $cb > 0$, there exists a critical value $k^* \in \mathbb{R}$, namely $k^* = a/cb$, such that, for each fixed $k > k^*$, $(a - kcb) < 0$, causing the system to be exponentially stable. In the case when the sign of cb is known, but not necessarily positive, noting that (1.2) takes the form

$$\dot{x}(t) = (a - bkc)x(t), \qquad x(0) = x^0 \in \mathbb{R},$$

with solution $x(t) = x^0 e^{(a-bkc)t}$, exponential stability follows provided that (i) k and cb are of the same sign and (ii) $a/|cb| < |k|$. More importantly, a great deal of research has been conducted into adaptive controllers for the case when a, b and c are unknown, subject to the restriction that $cb > 0$. Here, a function k is used in a feedback control given by the following:

$$u(t) = -k(t)y(t).$$

The intuition behind the controller is simple. By increasing $k(t)$, the gain will eventually be large enough so that $(a - k(t)cb) < 0$. The control should also be designed so that, as $y(t)$ approaches the origin, the rate of increase of the gain reduces. The adaptive control law

$$u(t) = -k(t)y(t), \quad \dot{k}(t) = y^2(t), \quad k(0) = k^0 \in \mathbb{R}$$

is implemented in [8] in order to achieve the following objectives:

(i) a solution $x(t)$ exists for all $t \in \mathbb{R}_+$,

(ii) the function k is bounded,

(iii) $x(t) \to 0$ as $t \to \infty$.

A disadvantage of the adaptive control scheme above is that the gain function k is non-decreasing. A key feature of the work later in this thesis is that the controllers are non-adaptive (in the sense that the gain is not generated by a differential equation) with gain functions that can potentially decrease.

1.4.2 Unknown sign high-frequency gain

In [49], Morse posed the problem of achieving the three objectives above for the case when the sign of cb is unknown and only the condition $cb \neq 0$ is imposed on the system (1.2)–(1.3). More precisely, differentiable functions ψ and u were sought with the property that, for all $cb \neq 0$ and $(x^0, k^0) \in \mathbb{R}^2$, the solution (x, k) of

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(cx(t), k(t)), \qquad & x(0) &= x^0 \in \mathbb{R}, \\ \dot{k}(t) &= \psi(cx(t), k(t)), & k(0) &= k^0 \end{aligned}$$

satisfies:

(i) a solution $x(t)$ exists for all $t \in \mathbb{R}_+$,

(ii) k is bounded,

(iii) $x(t) \to 0$ as $t \to \infty$.

Nussbaum [51] provided an answer to the problem with the following differentiable functions:

$$\begin{aligned} u &\colon (y, k) \mapsto (k^2 + 1)h(k)y, \\ \psi &\colon (y, k) \mapsto y(k^2 + 1), \end{aligned}$$

with the particular choice $h(k) = \cos(\frac{1}{2}\pi k)\exp(k^2)$ sufficing. Morse [50] then simplified this with the following control strategy:

$$\begin{aligned} u &: (y,k) \mapsto k^2\cos(k)y, \\ \psi &: (y,k) \mapsto y^2. \end{aligned}$$

Through the efforts of Willems and Byrnes [69], a more general control strategy, containing, as a special case, the controller of Morse [50] above, was proposed:

$$\left.\begin{aligned} u &: (y,k) \mapsto \nu(k)y, \\ \psi &: (y,k) \mapsto y^2, \end{aligned}\right\} \tag{1.4}$$

where the function $\nu\colon \mathbb{R} \to \mathbb{R}$ is bounded on compact sets and required to satisfy the properties

$$\limsup_{k\to\infty} \frac{1}{k}\int_0^k \nu(\kappa)\,\mathrm{d}\kappa = \infty, \qquad \liminf_{k\to\infty} \frac{1}{k}\int_0^k \nu(\kappa)\,\mathrm{d}\kappa = -\infty, \tag{1.5}$$

developed from the so-called Nussbaum conditions introduced in [51].

1.4.3 Multi-input, multi-output systems

A more general class of systems must be considered if even basic mechanical examples such as the pendulum (1.1) are to be incorporated. A generalization would be to consider single-input, single-output systems with state $x(t) \in \mathbb{R}^n$ and parameters $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^{1\times n}$, but we widen the scope further to include multi-input ($u(t) \in \mathbb{R}^m$), multi-output ($y(t) \in \mathbb{R}^m$) linear systems of the form

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = x^0 \in \mathbb{R}^n, \\ y(t) &= Cx(t), \end{aligned}\right\} \tag{1.6}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$.

Observe that, in the case of the basic prototype (1.1), setting $x_1 = y$ and $x_2 = \dot{y}$, the system can be rewritten as follows:

$$\begin{aligned} &\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} g\sin(x_1(t)) + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t), \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = x^0 \in \mathbb{R}^2, \\ &y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \end{aligned} \tag{1.7}$$

Clearly, the nonlinear term $g\sin(x_1(t))$ in (1.7) will necessitate the consideration of a more general class of systems than those of the form (1.6), see the investigations in Chapter 4. In order to handle multi-input, multi-output systems, several key structural assumptions are introduced.

1.4.4 Minimum-phase condition

We say that a multi-input, multi-output linear system of the form (1.6) is minimum-phase if the following condition holds:

$$\det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0 \qquad \forall s \in \overline{\mathbb{C}}_+. \tag{1.8}$$

Denote the transfer function $s \mapsto C(sI-A)^{-1}B$ of a linear system of the form (1.6) by $G \in \mathbb{R}(s)^{m\times m}$. For an output y produced by the system (1.6) starting at the zero initial state with input u, the relation $\hat{y}(s) = G(s)\hat{u}(s)$ holds between the Laplace transforms $\hat{y}$ and $\hat{u}$ of the output and input respectively. As described in [21, Definition 2.1.1], G is a rational matrix with *Smith-McMillan* form

$$\operatorname{diag}\left\{\frac{\varepsilon_1(s)}{\psi_1(s)},\ldots,\frac{\varepsilon_r(s)}{\psi_r(s)},0,\ldots,0\right\} = (U(s))^{-1}G(s)(V(s))^{-1},$$

where $U, V \in \mathbb{R}[s]^{m\times m}$ are unimodular, $\mathrm{rk}_{\mathbb{R}(s)}G = r$, $\varepsilon_i, \psi_i \in \mathbb{R}[s]$ are monic, coprime and satisfy $\varepsilon_i|\varepsilon_{i+1}$, $\psi_{i+1}|\psi_i$ for $i = 1,\ldots,r$. Setting

$$\varepsilon(s) := \prod_{i=1}^{r} \varepsilon_i(s), \qquad \psi(s) := \prod_{i=1}^{r} \psi_i(s),$$

a *zero* of G is a value s_0 such that $\varepsilon(s_0) = 0$ and s_0 is a *pole* of G if $\psi(s_0) = 0$. Coppel [11, Theorem 10] has shown (see also [21, Proposition 2.1.2]) that the minimum-phase condition is equivalent to: (A, B) is stabilizable (characterized by the existence of a matrix $F \in \mathbb{R}^{m\times n}$ such that $\sigma(A+BF) \subset \mathbb{C}_-$), (C, A) is detectable (the pair (A^T, C^T) is stabilizable), and the transfer function G has no zeros in the closed right half complex plane.

For convenience in later chapters, we introduce some notation. Define, by $\mathcal{L}$, the prototype class of finite-dimensional, minimum-phase, m-input, m-output linear systems (A, B, C) with sign-definite high-frequency gain, in the sense that either CB or $-CB$

is positive definite (symmetry of CB is not assumed). Specifically,

$$\mathcal{L} = \{(A, B, C) \mid A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m},\ C \in \mathbb{R}^{m\times n},\ m, n \in \mathbb{N},\\ CB \text{ sign definite, (1.8) holds}\}.$$

1.4.5 State space transformation

It has been shown in [21, Lemma 2.1.3], for example, that, provided $\det(CB) \neq 0$, the state space can be decomposed into the direct sum $\mathbb{R}^n = \operatorname{im} B \oplus \ker C$ and consequently systems of the form (1.6) can be rewritten as follows:

$$\left.\begin{aligned} \dot{y}(t) &= A_1 y(t) + A_2 z(t) + CBu(t), & y(0) &= y^0, \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t), & z(0) &= z^0, \end{aligned}\right\} \tag{1.9}$$

where $A_1 \in \mathbb{R}^{m\times m}$, $A_2 \in \mathbb{R}^{m\times(n-m)}$, $A_3 \in \mathbb{R}^{(n-m)\times m}$ and $A_4 \in \mathbb{R}^{(n-m)\times(n-m)}$. To see this, let $V \in \mathbb{R}^{n\times(n-m)}$ denote a basis of $\ker C$ and let $S := [B(CB)^{-1}, V]$, with inverse

$$S^{-1} = \begin{bmatrix} C \\ N \end{bmatrix}, \quad \text{where} \quad N = (V^T V)^{-1} V^T [I - B(CB)^{-1} C],$$

whence the similarity transformation

$$S^{-1}x = \begin{bmatrix} Cx \\ Nx \end{bmatrix} = \begin{bmatrix} y \\ z \end{bmatrix}, \ S^{-1}AS = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \ S^{-1}B = \begin{bmatrix} CB \\ 0 \end{bmatrix} \text{ and } CS = \begin{bmatrix} I & 0 \end{bmatrix}. \tag{1.10}$$

Importantly, if the triple (A, B, C) constitutes a minimum-phase system, it can be shown that A_4 is asymptotically stable. In other words, minimum-phase systems of the form (1.9) have stable zero-dynamics (the residual dynamics when the input u is such that the output y vanishes identically). Note that, in the case of the pendulum (1.7), $C = [I \quad 0]$ and $B = [0 \quad 1/m]^T$, so $CB = 0$ and hence the system cannot be rewritten in the form (1.9). In order to handle such systems, we first define what is meant by the relative degree of a linear system.

1.4.6 Relative degree

Consider a system of the form (1.6). We define the relative degree of (1.6), denoted ρ, for some $\rho \in \mathbb{N}$, by the property that $CA^iB = 0$ for $i = 1, \ldots, \rho - 2$ and $CA^{\rho-1}B \neq 0$. For example, the single-input, single-output system (1.2) with $cb \neq 0$ has relative degree $\rho = 1$. An alternative characterization of the relative degree of such systems, at the transfer function level, can be found in [21, Section 2.1].

Much of the literature in the field of high-gain feedback stabilization and tracking for systems of the form (1.6) imposes the condition that the relative degree must be 1, however, considerable attention has also been paid to the $\rho \geq 1$ case, see for example [48] and [33] among others. Here, we briefly discuss several notable papers that consider tracking and stabilization of high relative degree systems.

Bullinger and Allgöwer [5] introduce an observer in conjunction with an adaptive controller to achieve tracking with prescribed asymptotic accuracy $\lambda > 0$. This is achieved for a class of systems which are affine in the control, of known relative degree, and with affine, linearly bounded drift term. Ye [71] considers linear minimum-phase systems with nonlinear perturbation; the control objective is (continuous) adaptive λ-tracking with non-decreasing gain. Stabilization for systems of maximum relative degree in the so-called "parametric strict feedback form" is achieved in [72] via a piecewise constant adaptive switching strategy. Both these contributions use a backstepping procedure (an algorithm for feedback control synthesis that implements a Lyapunov style approach, see, for example, the description and basic example provided in [35]).

Note that the state space transformation used to interchange systems of the form (1.6) and (1.9) is not applicable to the case when $\rho > 1$. In Chapter 5, a state space transformation with similarities to (1.10) will be described for a class of systems with known relative degree $\rho \geq 1$.

1.4.7 Control techniques for multi-input, multi-output systems

The study of multi-input, multi-output systems, whilst more complicated than the single-input, single-output case, has developed in much the same way. Stabilization of the output y, whilst maintaining boundedness of the state x and gain function k, was achieved in [8] and, since then, many papers have investigated reference signal tracking for such systems. Setting

$$e(t) = y(t) - r(t), \quad e^0 = y^0 - r(0),$$

system (1.9) can be rewritten as follows:

$$\left.\begin{aligned} \dot{e}(t) &= A_1(e(t) + r(t)) + A_2 z(t) + CBu(t) - \dot{r}(t), & e(0) &= e^0, \\ \dot{z}(t) &= A_3(e(t) + r(t)) + A_4 z(t), & z(0) &= z^0. \end{aligned}\right\} \tag{1.11}$$

In [23], a class of systems of the form (1.6) is considered. Approximate tracking of every reference signal in the Sobolev space $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ is assured via the following controller

$$\left.\begin{aligned} &u(t) = -k(t)e(t), \\ &\dot{k}(t) = \begin{cases} (\|e(t)\| - \lambda)\|e(t)\|, & \text{if } \|e(t)\| \geq \lambda, \\ 0, & \text{if } \|e(t)\| < \lambda, \end{cases} \\ &k(0) = k^0 \in \mathbb{R}. \end{aligned}\right\} \tag{1.12}$$

The state x and associated gain function k remain bounded whilst the control strategy guarantees that the output error approaches the closed ball $\overline{\mathbb{B}}_\lambda$, that is, $\|e(t)\| \to [-\lambda, \lambda]$ as $t \to \infty$. In this sense, the output feedback strategy (1.12) is a λ-servomechanism. However, transient behaviour is not considered and, as mentioned in Section 1.2.4, this may lead to undesirable behaviour in the form of large output excursions.

The paper [48] considers the class of systems (1.6) with known relative degree, satisfying the minimum-phase assumption, restricted to the single-input, single-output case with high-frequency gain of known sign. Therein, a controller is introduced which guarantees the "error to be less than an (arbitrarily small) prespecified constant after an (arbitrarily small) prespecified period of time, with an (arbitrarily small) prespecified upper bound on the amount of overshoot."

The controller in [48] is less flexible in its scope for shaping transient behaviour when compared with the performance funnel approach in [26]. For example, an *a priori* bound on the initial data is required. However, this is counter-balanced by the fact that the class of systems in [26] is restricted to the relative degree 1 case. The controller in [26] takes the simple form

$$u(t) = -\alpha(\varphi(t)\|e(t)\|)e(t), \tag{1.13}$$

where $\alpha\colon [0,1) \to \mathbb{R}_+$ is a continuous, unbounded injection (for example $\alpha(s) = 1/(1-s)$). The intuition behind the control strategy is that, if the error $(e(t))$ approaches the funnel boundary, the gain $\alpha(\varphi(t)\|e(t)\|)$ increases which, in conjunction with a high-gain property of the system class, precludes boundary contact. In [26], both approximate tracking and stabilization are achieved with prescribed transient behaviour. In Chapter 4, the results on approximate tracking will be extended to the case when the relative degree is known, but not necessarily 1.

1.5 Infinite-dimensional linear systems

A great deal of work has gone into developing a mathematical framework which enables the generalization of the finite-dimensional results above to infinite dimensions.

The motivation behind this lies in the desire to model and control more complex systems including those described by partial differential equations (distributed parameter systems) and delay equations in which the state space is an infinite-dimensional function space. Background information on infinite-dimensional systems theory can be found in [13]. An overview of universal adaptive control techniques, in an infinite-dimensional setting, is provided in [43]. A basic example involving linear systems in infinite dimensions consisting of finite-dimensional input and output spaces and an infinite-dimensional Hilbert space X as the state space, is investigated in [57]. The class of nonlinear operators discussed in Chapter 2 will allow for interesting infinite-dimensional effects such as delays.

1.6 Nonlinear systems

Control objectives such as stabilization and reference signal tracking have also been considered in the context of nonlinear systems with greater generality than the prototypes discussed in Section 1.4. For instance, in [26], approximate tracking with prescribed transient behaviour is achieved for a class of multi-input, multi-output, infinite-dimensional nonlinear systems given by a functional differential equation of the form

$$\dot{y}(t) = g(p(t), (Ty)(t), u(t)), \quad y|_{[-h,0]} = y^0 \in C([-h,0], \mathbb{R}^m),$$

where g is a continuous function, p represents a bounded perturbation and T is a nonlinear, causal operator. Controller (1.13) is applied in conjunction with a performance funnel.

Jiang *et al.* [33] consider a large class of nonlinear systems which are single-input, single-output, have known relative degree and zero-dynamics which are stable in an appropriate sense. The emphasis therein lies on the nonlinear nature of the system class; neither tracking nor transient behaviour is addressed. Numerous other papers tackle nonlinear systems, see for example [54] and [56] which implement discontinuous feedback methods (see the discussion in Section 1.7.2). The adaptive results in [34] achieve stabilization of the output for a class of nonlinear systems via an adaptive strategy based on a high-gain compensator, but transient behaviour is not considered.

Non-adaptive contributions are found in the work by Byrnes and Isidori [6] with extensions in [7]. The two papers cover stabilization and tracking for a class of relative-degree-one nonlinear systems, with exogenous (disturbance) signals generated by an exosystem. The exosystem is subject to a *Poisson stability* assumption, by which it is meant that any point in the (compact and invariant) set of admissible initial conditions of the exosystem is an ω-limit point of a (potentially different) point in the same set. Systems of higher relative degree are also considered, see in particular [6, (33)], and the authors state (without proof) that "these systems can reduced to systems of (relative degree 1) by means of the semiglobal back-stepping lemma". The main result in [6, Proposition 7.1] pertains to practical tracking and applies high-gain principles in conjunction with an internal model (discussed below in Section 1.7.1). Related investigations, based on high-gain properties and/or an internal model principle, can be found in [37] and [52].

1.6.1 Class of nonlinear operators

To expand further the scope of investigations later in this thesis, a large class of nonlinear, causal operators, denoted $\mathcal{T}_h^m$, will be discussed in Chapter 2. The simple multi-input, multi-output linear system given by (1.9) will be used to provide insight into the inclusion of the operator class, though the main motivation will come from the wide range of hysteretic effects and other nonlinearities that the class of operators admits. The operators $\mathcal{T}_h^m$ will then be incorporated in the system classes discussed in Chapters 5, 6 and 7 as well as the existence theory developed in Chapter 3.

1.7 Control methods

As stated in Section 1.1, the control strategies developed in this thesis are universal. This means that, for a class of systems satisfying structural assumptions such as the ones discussed above in Section 1.4.3, a control strategy is devised that is capable of achieving the control objectives for any system in the class.

Where possible, continuous feedback controllers will be constructed, akin to those mentioned already, see (1.4) and (1.13) for example. However, two alternative control approaches are discussed below. The first method involves an internal model.

1.7.1 Internal models

The Internal Model Principle, see Wonham [70, Section 8.8], states that:

"A regulator is structurally stable only if the controller utilizes feedback of the regulated variable and incorporates in the feedback loop a suitably reduplicated model of the dynamic structure of the exogenous signals which the regulator is required to process."

Wonham adds that, loosely speaking, the principle states:

"every good regulator must incorporate a model of the outside world."

The idea is to include an internal model, capable of generating the class of reference signals to be tracked, in series interconnection with a feedback controller, as illustrated in Figure 1-5. An internal model, applied in series with an adaptive stabilizer, was implemented in [21, Section 5.1].

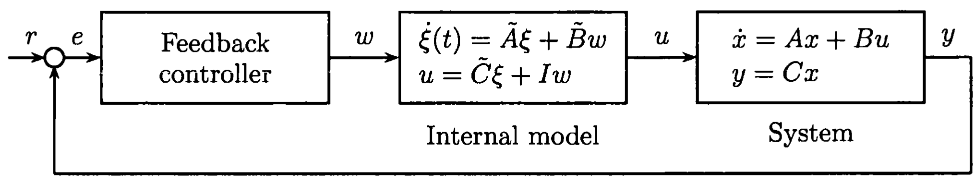


Figure 1-5: Feedback control with an internal model.

The internal model approach has more recently been investigated in conjunction with a performance funnel in [24] for a class of linear systems of the form (1.6), with relative degree 1 and sign-definite high-frequency gain, satisfying the minimum-phase assumption. A controller was developed with an internal model to ensure prescribed transient behaviour and an asymptotic tracking objective were achieved. Due to the linear nature of the internal model, the class of reference signals considered is restrictive.

The controllers implemented in this thesis require no explicit knowledge of the system beyond basic structural assumptions. However, comparisons will be made with the above internal model approach.

1.7.2 Discontinuous control

Recall that, paraphrasing W M Wonham [70, Page 210], the internal model principle states that every "good" regulator must incorporate a model of the outside world. The principle may appear to suggest that, in order to achieve exact asymptotic tracking of a particular class of reference signals via continuous feedback control, an internal model, capable of replicating the signals to be tracked, is required. We will see in Chapter 6 that this may not always be the case and a continuous controller capable of ensuring exact asymptotic tracking will be constructed. However, in the context of linear systems with linear regulators (see, [70, 67]), "good" means "structurally stable"; in a more general context of smooth nonlinear systems (see, [62]), "good" amounts to a "signal detection" property. In effect, "good" implies some robustness property of the closed loop. The feedback structure that will be proposed in Chapter 6 is designed to ensure tracking of any signal of class $W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^m)$, yet it does not contain a model capable of replicating this class of signals. For consistency with the internal model principle, one must therefore conclude that the closed-loop system of Section 6.5.1 lacks some robustness property.

The full generality of the control strategy developed in Chapter 6 does also, however, encompass potentially discontinuous feedback controllers. It is well known that discontinuous feedback control can be used to achieve exact asymptotic tracking. In [55], for example, the system class comprises nonlinearly-perturbed linear systems satisfying standard assumptions such as the minimum-phase condition and appropriate bounds on the nonlinearities. Exact asymptotic tracking of all reference functions on the Sobolev space $W^{1,\infty}(\mathbb{R},\mathbb{R}^m)$ is achieved via a discontinuous control which, for the purposes of building a suitable existence theory for the system, is interpreted in a set-valued sense.

The use of a discontinuous controller in Chapter 6 and the presence of a large class of nonlinear operators, discussed in Chapter 2, motivate the construction of a suitable existence theory in Chapter 3. The main result of the chapter will prove the existence of a maximal solution to a class of functional differential inclusions and is preceded by an existence theorem proving the existence of a solution in the restricted case of functional differential equations.

1.8 Applications

The main results of this thesis can be viewed as contributions which are analytical in nature, addressing the question of existence of controllers which guarantee the two main performance objectives under weak hypotheses. The high-gain feedback controllers implemented here may be criticized, particularly in the cases when the direction of the controller is unknown (and the techniques discussed in Section 1.4.2 are applied), for the fact that, despite the input signal remaining bounded, control values could grow too large to be feasible in many practical situations.

However, the work by Ilchmann and Trenn in [30] demonstrates the application of a performance funnel controller, subject to input constraints, to a model for chemical reactors. This illustrates that, in some situations, when the focus is shifted to controller synthesis, practical applications may be possible. More recently, a performance funnel, in conjunction with a proportional-integral controller and a high-pass filter, has been applied to a two mass system modelled by functional differential equations, see [29]; additionally, the controller is implemented on a real plant, an electrical drive.

1.9 Thesis overview

We follow this chapter with the introduction of a class of nonlinear operators $\mathcal{T}_h^m$, in Chapter 2, and provide a discussion of diverse phenomena, such as delays and hysteresis, that can be incorporated in $\mathcal{T}_h^m$. In Chapter 3, a suitable existence theory is developed for systems of functional differential equations and inclusions involving the operators $T \in \mathcal{T}_h^m$.

Chapter 4 examines the asymptotic and transient behaviour of a nonlinearly-perturbed class of multi-input, multi-output, linear systems of known relative degree; a continuous feedback strategy is implemented and an approximate tracking objective is sought. In Chapter 5 the class of systems considered is expanded to a large class of nonlinear systems and a continuous feedback strategy is implemented in order to achieve approximate tracking.

The last two areas of research, in Chapters 6 and 7, restrict attention to systems of relative degree one, but this limitation is compensated for by targeting an exact asymptotic tracking result. The first investigation, in Chapter 6, involves a potentially discontinuous feedback controller applied to a class of multi-input, multi-output nonlinear systems. Asymptotic tracking and approximate tracking will be developed in unison within a framework of functional differential inclusions, making use of the existence theory developed in Chapter 3. Finally, Chapter 7 examines a class of single-input, single-output, nonlinear systems subject to input hysteresis and a continuous controller is implemented.

Following the completion of the main results in this thesis, there is a short section containing several concluding remarks, followed by an appendix involving technical results and a few basic concepts designed to make this thesis relatively self-contained.

Chapter 2

Class of nonlinear operators

In Chapter 4, nonlinearly-perturbed, multi-input, multi-output, linear systems will be considered whilst, in Chapter 5, the class of systems will be expanded to encompass a wider range of nonlinear systems described by functional differential equations. Inherent in these are causal operators and so, in anticipation of the latter investigation, we begin by introducing a class of nonlinear operators $\mathcal{T}_h^m$ that will play a central role in Chapters 5, 6 and 7.

The first section of this Chapter provides the definition of the operator class and several remarks on the properties of the operators. This is followed, in Section 2.2, by a discussion of delays and hysteretic effects encapsulated by operators $T \in \mathcal{T}_m^h$. The inclusion of these nonlinear operators in the systems considered in later chapters motivates the development of a suitable existence theory for functional differential equations and inclusions in Chapter 3.

2.1 Class of operators

Fix $m \in \mathbb{N}$ arbitrarily. For later convenience, we introduce some notation: for $h, t \in \mathbb{R}_+$, $w \in C([-h,t],\mathbb{R}^m)$, $\tau > t$ and $\delta > 0$, define

$$\mathcal{C}(w;h,t,\tau,\delta) := \{v \in C([-h,\tau],\mathbb{R}^m) \mid \; v|_{[-h,t]} = w, \; \|v(s) - w(t)\| \le \delta \;\; \forall\, s \in [t,\tau]\}.$$

Definition 2.1.1 (Operator class $\mathcal{T}_h^m$)
An operator T is said to be of class $\mathcal{T}_h^m$ if, and only if, the following hold.

(i) *For some $q \in \mathbb{N}$, $T\colon C([-h,\infty),\mathbb{R}^m) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^q)$.*

(ii) *For all* $t \geq 0$ *and all* $x, y \in C([-h,\infty),\mathbb{R}^m)$,

$$x(\cdot) \equiv y(\cdot) \text{ on } [-h,t] \quad \Longrightarrow \quad (Tx)(s) = (Ty)(s) \text{ for almost all } s \in [0,t].$$

(iii) *For each* $t \geq 0$ *and each* $w \in C([-h,t],\mathbb{R}^m)$, *there exist* $\tau > t$, $\delta > 0$ *and* $c_0 > 0$ *such that, for all* $x, y \in \mathcal{C}(w; h, t, \tau, \delta)$,

$$\text{ess-sup}_{s\in[t,\tau]}\|(Tx)(s) - (Ty)(s)\| \leq c_0 \sup_{s\in[t,\tau]} \|x(s) - y(s)\|.$$

(iv) *For every* $c_1 > 0$, *there exists* $c_2 > 0$ *such that, for all* $y \in C([-h,\infty),\mathbb{R}^m)$,

$$\sup_{t\in[-h,\infty)} \|y(t)\| \leq c_1 \quad \Longrightarrow \quad \|(Ty)(t)\| \leq c_2 \quad \text{for almost all } t \geq 0\,.$$

Remark 2.1.2

(i) Property (ii) is a natural assumption of causality. Property (iv) is a bounded-input, bounded-output assumption on the operator T.

(ii) Property (iii) is a technical assumption of local Lipschitz type which is used in establishing well-posedness of the closed-loop systems considered in later chapters. To interpret (iii) correctly, we need to give meaning to Tx for a function $x \in C(I,\mathbb{R}^m)$ on a bounded interval I of the form $[-h,a)$ or $[-h,a]$, where $0 < a < \infty$. This we do by showing that T "localizes", in a natural way, to an operator $\tilde{T}\colon C(I,\mathbb{R}^m) \to L^\infty_{\text{loc}}(J,\mathbb{R}^q)$, where $J := I \setminus [-h,0)$. Let $x \in C(I,\mathbb{R}^m)$. For each $\sigma \in J$, define $x_\sigma \in C([-h,\infty),\mathbb{R}^m)$ by

$$x_\sigma(t) := \begin{cases} x(t), & t \in [-h,\sigma], \\ x(\sigma), & t > \sigma\,. \end{cases}$$

By causality, we may define $\tilde{T}x \in L^\infty_{\text{loc}}(J,\mathbb{R}^q)$ by the property

$$\tilde{T}x|_{[0,\sigma]} = Tx_\sigma|_{[0,\sigma]} \quad \forall\, \sigma \in J.$$

Henceforth, we will not distinguish notationally an operator T and its "localisation" $\tilde{T}$: the correct interpretation being clear from context.

The following properties of the operator class $\mathcal{T}^m_h$, proved in [59], are assembled here for convenience and subsequently used without further comment.

(a) $\mathcal{T}_h^m$ is a linear space:

$$T_1, T_2 \in \mathcal{T}_h^m \Longrightarrow (a_1 T_1 + a_2 T_2) \in \mathcal{T}_h^m \quad \forall a_1, a_2 \in \mathbb{R}.$$

(b) If $h_1 < h_2$, then $\mathcal{T}_{h_1}^m \subset \mathcal{T}_{h_2}^m$ in the sense that

$$T \in \mathcal{T}_{h_1}^m \Longrightarrow \hat{T} \in \mathcal{T}_{h_2}^m,$$

where $(\hat{T}y)(t) := (Ty|_{[-h_1,\infty)})(t)$ for all $t \in \mathbb{R}_+$ and $y \in C([-h_2,\infty),\mathbb{R}^m)$.

(c) For all $r \in W^{1,\infty}(\mathbb{R},\mathbb{R}^m)$, $T \in \mathcal{T}_h^m$ implies that $T_r \in \mathcal{T}_h^m$, where T_r is given by

$$(T_r y)(t) := (T(y+r))(t) \quad \forall t > 0, \quad y \in C([-h,\infty),\mathbb{R}^m).$$

2.2 Examples of hysteresis and nonlinearities in $\mathcal{T}_h^m$

In this section, several interesting phenomena encompassed by the operator class $\mathcal{T}_h^m$ are highlighted. We begin with the simple prototype class of multi-input, multi-output, linear systems.

2.2.1 Multi-input, multi-output, linear systems

Consider the class $\mathcal{L}$ of finite-dimensional, minimum-phase, m-input, m-output linear systems (A, B, C) with sign-definite high-frequency gain. Recall from Section 1.4.5 that, following an appropriate similarity transform, such systems can be rewritten as

$$\left.\begin{array}{ll} \dot{y}(t) = A_1 y(t) + A_2 z(t) + CBu(t), & y(0) = y^0, \\ \dot{z}(t) = A_3 y(t) + A_4 z(t), & z(0) = z^0, \end{array}\right\} \tag{2.1}$$

where, by the minimum-phase property, A_4 is Hurwitz. Defining the function d (continuous and bounded) and operator T (linear) by

$$\left.\begin{array}{rcl} d(t) & := & A_2\big(\exp(A_4 t)\big)z^0, \\ (Ty)(t) & := & A_1 y(t) + A_2 \int_0^t \big(\exp A_4(t-s)\big) A_3 y(s)ds, \end{array}\right\} \tag{2.2}$$

we see that the original system $(A, B, C) \in \mathcal{L}$ can be recast in the form of the following (linear) functional differential equation.

$$\dot{y}(t) = d(t) + (Ty)(t) + CB\,u(t), \quad y(0) = y^0 \in \mathbb{R}^m. \tag{2.3}$$

The operator T, so defined, is clearly of class $\mathcal{T}_0^m$. We will make use of this basic prototype in Chapter 6.

2.2.2 Input-to-state stable systems

Let $g\colon \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^p$ be locally Lipschitz with $g(0,0) = 0$. For $y \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$, let $z(\cdot, z^0, y)$ denote the unique maximal solution of the initial-value problem

$$\dot{z}(t) = g(z(t), y(t)), \quad z(0) = z^0 \in \mathbb{R}^p. \tag{2.4}$$

Assume that system (2.4) is input-to-state stable (ISS), that is, there exist functions $\theta \in KL$ and $\gamma \in K_\infty$ such that, for all $(z^0, y) \in \mathbb{R}^p \times L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$,

$$\|z(t, z^0, y)\| \le \theta(\|z^0\|, t) + \operatorname*{ess\text{-}sup}_{s\in[0,t]} \gamma(\|y(s)\|) \quad \forall t \ge 0, \tag{2.5}$$

see [60] and more recent papers such as [61], [64], and [1], and references therein, for a detailed treatment of ISS systems.

Let $W\colon \mathbb{R}^p \to \mathbb{R}^m$ be a locally Lipschitz function with the property that there exists $L > 0$ such that $\|W(z)\| \le L\|z\|$ for all z. Therefore, assuming that system (2.4) has output given by $W(z(t, z^0, y))$ and fixing $z^0 \in \mathbb{R}^p$ arbitrarily, we may define an operator $T\colon C(\mathbb{R}_+, \mathbb{R}^m) \to L^\infty_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^m)$ by

$$(Ty)(t) := W(z(t, z^0, y)). \tag{2.6}$$

Proposition 2.2.1 *The operator* (2.6) *is of class* $\mathcal{T}_0^m$.

Proof. Observe that, in view of the input-to-state stability property (2.5) and the properties of W, there exists $c > 0$ such that, for all $y \in C(\mathbb{R}_+, \mathbb{R}^m)$,

$$\|(Ty)(s)\| \le c\left[1 + \sup_{\tau\in[0,s]} \gamma(\|y(\tau)\|)\right] \quad \forall s \ge 0.$$

Assumptions (i), (ii) and (iv) of the class $\mathcal{T}_h^m$ clearly hold. To establish (iii), we proceed as follows. Let $t \ge 0$, $\rho > 0$ and $\zeta \in C([0,t], \mathbb{R}^m)$. Let $r > 0$ and define $R := \sup_{\tau\in[0,t]} \|\zeta(\tau)\| + r$. By input-to-state stability there exists a compact set $K \subset \mathbb{R}^p$ such that, for all y with $\sup_{\tau\in\mathbb{R}_+} \|y(\tau)\| \le R$ we have $z(s, z^0, y) \in K$ for all $s \ge 0$. Let $\lambda > 0$ be a Lipschitz constant for $g(\cdot,\cdot)$ on the set $K \times \mathbb{B}_R$. For all $y, x \in C(\mathbb{R}_+, \mathbb{R}^m)$

with $y|_{[0,t]} = \zeta = x|_{[0,t]}$ and $\|y(s)\|$, $\|x(s)\| \le R$ for almost all $s \in [0, t+\rho]$,

$$\|z(s, z^0, y) - z(s, z^0, x)\| \le \int_0^s \|g(z(\tau, z^0, y), y(\tau)) - g(z(\tau, z^0, x), x(\tau))\| d\tau$$
$$\le \lambda \int_t^s \big[\|z(\tau, z^0, y) - z(\tau, z^0, x)\| + \|x(\tau) - y(\tau)\|\big] d\tau$$

for all $s \in [t, t+\rho]$. By a version of Gronwall's Lemma (see Appendix A) it follows that

$$\|z(s, z^0, y) - z(s, z^0, x)\| \le \lambda \int_t^s \exp(\lambda(s-\tau))\|y(\tau) - x(\tau)\| ds \quad \forall s \in [t, t+\rho].$$

We may now conclude that there exists a constant $c_R > 0$ such that, for all $y, x \in C(\mathbb{R}_+, \mathbb{R}^m)$ with $y|_{[0,t]} = \zeta = x|_{[0,t]}$ and $\|y(s)\|$, $\|x(s)\| \le R$ for almost all $s \in [t, t+\rho]$,

$$\|(Ty)(s) - (Tx)(s)\| \le c_R \sup_{s \in [t, t+\rho]} \|y(s) - x(s)\| \quad \forall s \in [t, t+\rho]$$

and so Property (iii) of $\mathcal{T}_h^m$ holds. □

We will make use of this fact in an example in Chapter 6.

2.2.3 Nonlinear delay systems

Let functions $\mathcal{G}_i : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^q$, $(t, y) \mapsto \mathcal{G}_i(t, y)$, $i = 0, \ldots, n$ be measurable in t and locally Lipschitz in y uniformly with respect to t: precisely,

(G1) for each fixed y, $\mathcal{G}_i(\cdot, y)$ is measurable,

(G2) for every compact $\mathcal{K} \subset \mathbb{R}^m$ there exists a constant c_0 such that

$$\|\mathcal{G}_i(t, y) - \mathcal{G}_i(t, x)\| \le c\,\|y - x\| \quad \text{for a.a. } t \quad \forall y, x \in \mathcal{K}.$$

For $i = 0, \ldots, n$, let $h_i \in \mathbb{R}_+$ and define $h := \max_i h_i$. For $y \in C([-h, \infty), \mathbb{R}^m)$, let

$$(Ty)(t) := \int_{-h_0}^0 \mathcal{G}_0(s, y(t+s))\ \mathrm{d}s + \sum_{i=1}^n \mathcal{G}_i(t, y(t-h_i)) \quad \text{for all } t \ge 0. \tag{2.7}$$

Proposition 2.2.2 *The operator* (2.7) *is of class* $\mathcal{T}_h^m$.

Proof. We consider, separately, the cases of point and distributed delays.

(i) Point delay. Let $\mathcal{G} : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^q$ satisfy (G1) and (G2). Define $\hat{T}$ by

$$(\hat{T}y)(t) := \mathcal{G}(t, y(t-h)),$$

then, for $t \in \mathbb{R}_+$, $(\hat{T}y)(t)$ is well defined and $\hat{T}\colon C([-h,\infty),\mathbb{R}^m) \to L^\infty_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^q)$. Property (ii) of class $\mathcal{T}_h^m$ clearly holds, whilst (G1) and (G2) are sufficient to show that Properties (iii) and (iv) hold.

(ii) Distributed delay. Let $\mathcal{G}\colon \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^q$ satisfy (G1) and (G2). Define $\hat{T}$ by

$$(\hat{T}y)(t) := \int_{-h}^{0} \mathcal{G}(s, y(t+s))ds.$$

Clearly, Properties (i) and (ii) of class $\mathcal{T}_h^m$ hold. Let $\delta > 0$, I be a bounded interval and let $t^* := \sup\{t \mid t \in I\}$, then properties (G1) and (G2) of $\mathcal{G}$ ensure the existence of a constant c_1 such that

$$\text{for a.a. } t \in [-h, t^*] \quad \|\mathcal{G}(t,y)\| \le c_1 \quad \forall \|y\| < \delta.$$

Then, for all $t \in I$ and all $y \in C([-h,\infty),\mathbb{R}^m)$ with $\sup_{t\in[-h,\infty)}\|y(t)\| < \delta$,

$$\|(\hat{T}y)(t)\| \le \int_{-h}^{0} \|\mathcal{G}(s,y(t+s))\|ds \le h \operatorname*{ess-sup}_{s\in[-h,0]} \|\mathcal{G}(s,y(t+s))\| \le hc_1,$$

thus ensuring that Property (iv) of $\mathcal{T}_h^m$ is satisfied. It remains to prove that Property (iii) holds. Let $t \ge 0$, $\tau > t$, $\delta > 0$ and $w \in C([-h,t],\mathbb{R}^m)$. Let $y, x \in \mathcal{C}(w; h, t, \tau, \delta)$, then, by (G2), there exists a constant $c_0 > 0$ such that

$$\|\mathcal{G}(s,y(\sigma)) - \mathcal{G}(s,x(\sigma))\| \le c_0\|y(\sigma) - x(\sigma)\| \quad \text{for a.a. } s,\sigma \in [-h,\tau].$$

Then, for $s \in [t,\tau]$,

$$\begin{aligned}
\|(\hat{T}y)(s) - (\hat{T}x)(s)\| &\le \int_{-h}^{0} \|\mathcal{G}(\sigma, y(s+\sigma)) - \mathcal{G}(\sigma, x(s+\sigma))\|d\sigma \\
&\le h \operatorname*{ess-sup}_{\sigma\in[-h,0]} \|\mathcal{G}(\sigma, y(s+\sigma)) - \mathcal{G}(\sigma, x(s+\sigma))\| \\
&\le hc_0 \sup_{s\in[-h,0]} \|y(s+\sigma) - x(s+\sigma)\|
\end{aligned}$$

and hence

$$\begin{aligned}
\operatorname*{ess-sup}_{s\in[t,\tau]} \|(\hat{T}y)(s) - (\hat{T}x)(s)\| &\le hc_0 \sup_{s\in[t-h,\tau]} \|y(s) - x(s)\| \\
&= hc_0 \sup_{s\in[-h,\tau]} \|y(s) - x(s)\|.
\end{aligned}$$

Therefore, $\hat{T} \in \mathcal{T}_h^m$. □

2.2.4 Hysteresis operators

Hysteresis is a property of systems that do not react instantly to the forces applied to them or do not return completely to their original state. For example, hysteresis phenomena can be used to describe the elastic and electromagnetic behaviour of various materials in which a delay occurs between the application and removal of a force or field and its subsequent effect. Other applications can also be found in economics and biology. The word hysteresis is derived from the ancient Greek word for "deficiency". The term was coined by Sir James Alfred Ewing. We continue this section by mathematically defining what we mean by a *hysteresis operator.*

A function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a time transformation if it is continuous, non-decreasing and surjective. An operator $\Phi\colon C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ is rate-independent if, for every time transformation f,

$$(\Phi(y \circ f))(t) = (\Phi y)(f(t)) \quad \forall y \in C(\mathbb{R}_+,\mathbb{R})\ \forall t \in \mathbb{R}_+.$$

Definition 2.2.3 $\Phi\colon C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ *is a hysteresis operator if* Φ *is causal and rate independent.*

The numerical value set NVS Φ of a hysteresis operator Φ is defined by

$$\mathrm{NVS}\,\Phi := \{(\Phi y)(t) \mid y \in C(\mathbb{R}_+,\mathbb{R}),\ t \in \mathbb{R}_+\}.$$

A function $y \in C(\mathbb{R}_+,\mathbb{R})$ is called *ultimately non-decreasing* (*non-increasing*) if there exists $\tau \in \mathbb{R}_+$ such that y is non-decreasing (non-increasing) on $[\tau,\infty)$; y is said to be *approximately ultimately non-decreasing* (*non-increasing*), if, for all $\varepsilon > 0$ there exists an ultimately non-decreasing (non-increasing) function $x \in C(\mathbb{R}_+,\mathbb{R})$ such that

$$|y(t) - x(t)| \leq \varepsilon \quad \forall t \in \mathbb{R}_+.$$

In [40], a general class of hysteresis operators $\Phi\colon C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ is considered, satisfying the following assumptions:

(N1) Φ is causal,

(N2) for all $y \in C(\mathbb{R}_+,\mathbb{R})$ and all $\sigma \in \mathbb{R}_+$, $(\Phi y_\sigma)(t) = (\Phi y_\sigma)(\sigma)$ for all $t \geq \sigma$ (recall the definition of y_σ from Remark 2.1.2(ii) in the case $h = 0$),

(N3) $\Phi(AC(\mathbb{R}_+,\mathbb{R})) \subset AC(\mathbb{R}_+,\mathbb{R})$, (N4) Φ is monotone in the sense that, for all $y \in AC(\mathbb{R}_+, \mathbb{R})$ with $\Phi y \in AC(\mathbb{R}_+, \mathbb{R})$,

$$(\Phi y)'(t)\dot{y}(t) \geq 0 \quad \text{for a.a. } t \in \mathbb{R}_+,$$

(N5) for each $t \geq 0$ and $w \in C([0,t], \mathbb{R})$ there exist $\tau > t$, $\lambda > 0$ and $\delta > 0$ such that, for all $y, x \in \mathcal{C}(w; 0, t, \tau, \delta)$,

$$\sup_{s\in[t,\tau]} |(\Phi y)(s) - (\Phi x)(s)| \leq \lambda \sup_{s\in[t,\tau]} |y(s) - x(s)|,$$

(N6) if $y \in C(\mathbb{R}_+, \mathbb{R})$ is approximately ultimately nondecreasing and, furthermore, $\lim_{t\to\infty} y(t) = \infty$, then $(\Phi y)(t)$ converges to $\sup \mathrm{NVS}\,\Phi$ as $t \to \infty$ and $(\Phi(-y))(t)$ converges to $\inf \mathrm{NVS}\,\Phi$ as $t \to \infty$,

(N7) if $y \in C(\mathbb{R}_+, \mathbb{R})$ is such that $\lim_{t\to\infty}(\Phi y)(t) \in \mathrm{intNVS}\,\Phi$, then y is bounded,

(N8) for all $\tau > 0$ and all $y \in C([0,\tau), \mathbb{R})$, there exist $a, b > 0$ such that

$$\sup_{s\in[0,t]} |(\Phi y)(s)| \leq a + b \sup_{s\in[0,t]} |y(s)| \quad \forall t \in [0,\tau).$$

As discussed in [40], the operators satisfying (N1)–(N8) can take the form of many physically motivated hysteretic effects, examples of which include backlash hysteresis, elastic-plastic hysteresis and Preisach operators. In [25], it is demonstrated that the operators satisfying assumptions (N1)–(N8), are of class $\mathcal{T}_0^1$. For illustration, we describe four particular examples of a hysteresis operator, namely relay, backlash and elastic-plastic hysteresis and the rather more general Preisach operator.

Relay hysteresis

One of the more commonly referred to types of hysteresis is relay hysteresis, see for example [36], [40] and [44]. Let $a_1 < a_2$ and let $\rho_1 \colon [a_1, \infty) \to \mathbb{R}$, $\rho_2 \colon (-\infty, a_2] \to \mathbb{R}$ be continuous, globally Lipschitz and satisfy $\rho_1(a_1) = \rho_2(a_1)$ and $\rho_1(a_2) = \rho_2(a_2)$. For a given input $y \in C(\mathbb{R}_+, \mathbb{R})$ to the hysteresis element, the output $w(t) = (\mathcal{R}y)(t)$ is such that $(y(t), w(t)) \in \mathrm{graph}(\rho_1) \cup \mathrm{graph}(\rho_2)$ for all $t \in \mathbb{R}_+$: the value $w(t)$ of the output at $t \in \mathbb{R}_+$ is either $\rho_1(y(t))$ or $\rho_2(y(t))$, depending on which of the threshold values a_2 or a_1 was "last" attained by the input y.
For $y \in C(\mathbb{R}_+, \mathbb{R})$ and $t \geq 0$, define

$$S(y,t) := y^{-1}(\{a_1, a_2\}) \cap [0,t], \quad \tau(y,t) := \begin{cases} \max S(y,t) & \text{if } S(y,t) \neq \emptyset, \\ -1 & \text{if } S(y,t) = \emptyset. \end{cases}$$

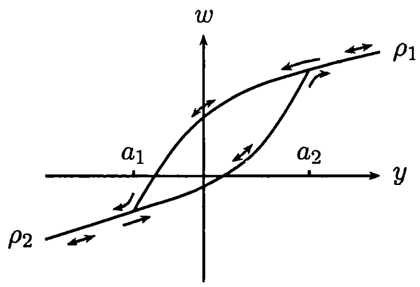


Figure 2-1: Relay hysteresis.

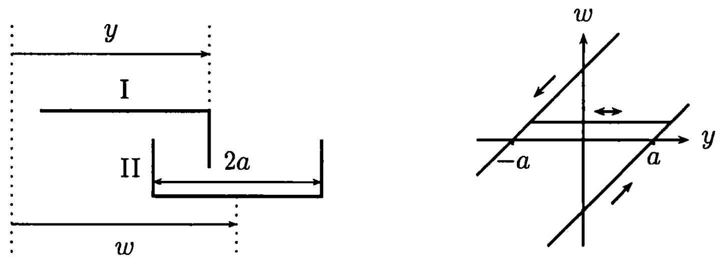
We now make precise the definition of a relay hysteresis operator. For each $\xi \in \mathbb{R}$, define the operator $\mathcal{R}_\xi \colon C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ by

$$(\mathcal{R}_\xi y)(t) = \begin{cases} \rho_2(y(t)) & \text{if } y(t) \leq a_1, \\ \rho_1(y(t)) & \text{if } y(t) \geq a_2, \\ \rho_2(y(t)) & \text{if } y(t) \in (a_1, a_2),\ \tau(y,t) \neq -1,\ y(\tau(y,t)) = a_1, \\ \rho_1(y(t)) & \text{if } y(t) \in (a_1, a_2),\ \tau(y,t) \neq -1,\ y(\tau(y,t)) = a_2, \\ \rho_2(y(t)) & \text{if } y(t) \in (a_1, a_2),\ \tau(y,t) = -1,\ \xi \leq 0, \\ \rho_1(y(t)) & \text{if } y(t) \in (a_1, a_2),\ \tau(y,t) = -1,\ \xi > 0. \end{cases}$$

The operator, so defined, is of class $\mathcal{T}_0^1$, (it is shown in [40, Section 5] that $\mathcal{R}_\xi$ satisfies assumptions (N1)–(N8)). A relay hysteresis operator is illustrated in Figure 2-1.

Backlash hysteresis

The backlash operator, sometimes referred to as a play operator, is discussed in detail in [4], [36] and [68]. Intuitively, the operator describes the input-output behaviour of mechanical play between two elements. Consider a one-dimensional mechanical link consisting of two components, denoted I and II (of width $2a$) and illustrated in Figure 2-2(a). The displacements of each part (with respect to some fixed datum) at time $t \geq 0$ are given by $y(t)$ and $w(t)$ with $|y(t) - w(t)| \leq a$ for all t, and $w(0) := y(0) + b$ for some pre-specified $b \in [-a, a]$. Within the link there is mechanical play: that is to say the position $w(t)$ of II remains constant as long as the position $y(t)$ of I remains



(a) (b)

Figure 2-2: Backlash hysteresis.

within the interior of II. Thus, assuming continuity of y, we have $\dot{w}(t) = 0$ whenever $|y(t) - w(t)| < a$. For a given input $y \in C(\mathbb{R}_+, \mathbb{R})$, describing the evolution of the position of I, denote the corresponding position of II by the output $w(t) = (\mathcal{B}y)(t)$

With a view to giving a precise definition of the backlash operator, we first define, for each $\mu \in \mathbb{R}_+$, the function $b_\mu \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$b_\mu(y, x) := \max\{y - \mu, \min\{y + \mu, x\}\}.$$

For all $\mu \in \mathbb{R}_+$ and all $\xi \in \mathbb{R}$, we introduce an operator $\mathcal{B}_{\mu,\xi}$ defined on the space $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ of piecewise monotone functions, by defining, for every $y \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$,

$$(\mathcal{B}_{\mu,\xi}(y))(t) = \begin{cases} b_\mu(y(0), \xi), & \text{for } t = 0, \\ b_\mu(y(t), (\mathcal{B}_{\mu,\xi}(y))(t_i)), & \text{for } t_{i-1} < t \le t_i,\ i \in \mathbb{N}, \end{cases}$$

where $0 = t_0 < t_1 < t_2 < \ldots$ is a partition of $\mathbb{R}_+$ such that y is monotone on each of the intervals $[t_{i-1}, t_i]$. Note that the definition is independent of the choice of partition. It is well known that the operator $\mathcal{B}_{\mu,\xi} \colon C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ can be extended uniquely to a hysteresis operator $\mathcal{B}_{\mu,\xi} \colon C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$, see, for example, [4]. The extended operator is of class $\mathcal{T}_0^1$ (for details, see [40]).

Elastic-plastic hysteresis

The elastic-plastic operator, sometimes called a stop operator, has much in common with the backlash operator and models the relationship of stress and strain in a one-dimensional elastic-plastic element. Provided that the stress (w) has magnitude less than the yield stress (μ), the strain y is related to w through the linear Hooke's Law. The stress, upon exceeding the yield value, remains constant under any increases in strain, though the elastic behaviour is instantly recovered when the strain is decreased. The formal definition, as provided in [4] and [40], follows. First, define, for each $\mu \in \mathbb{R}_+$, the function $e_\mu \colon \mathbb{R} \to \mathbb{R}$ by

$$e_\mu(y) = \min\{\mu, \max\{-\mu, y\}\}.$$

For all $\mu \in \mathbb{R}_+$ and all $\xi \in \mathbb{R}$, we introduce an operator $\mathcal{E}_{\mu,\xi}$ on $C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$ by defining recursively, for every $y \in C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R})$,

$$(\mathcal{E}_{\mu,\xi}(y))(t) = \begin{cases} e_\mu(y(0) - \xi), & \text{for } t = 0, \\ e_\mu(y(t) - y(t_i) + (\mathcal{E}_{\mu,\xi}(y))(t_i)), & \text{for } t_{i-1} < t \le t_i,\ i \in \mathbb{N}, \end{cases}$$

where $0 = t_0 < t_1 < t_2 < \ldots$ is a partition of $\mathbb{R}_+$ such that y is monotone on each of the intervals $[t_{i-1}, t_i]$. Note that, as was the case with the backlash operator, the definition is independent of the choice of partition. It is shown in [40] that $\mathcal{E}_{\mu,\xi} \colon C_{\mathrm{pm}}(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ can be extended uniquely to a hysteresis operator $\mathcal{E}_{\mu,\xi} \colon C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ which is of class $\mathcal{T}_0^1$ (see [40, Proposition 14.5]). The action of the elastic plastic operator is illustrated in Figure 2-3.

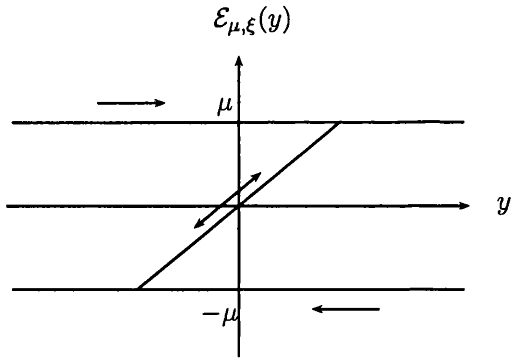


Figure 2-3: Elastic-plastic hysteresis.

Preisach and Prandtl operators

A more general type of operator in $\mathcal{T}_h^m$ is the Preisach operator, encompassing backlash and elastic-plastic hysteresis as well as Prandtl operators. An interesting feature of Preisach operators is that the hysteresis action, for certain input functions, exhibits nested loops in the corresponding input-output characteresistics. Below, we describe both the Preisach and Prandtl operator in more detail.

Let $\zeta \colon \mathbb{R}_+ \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Let μ be a signed Borel measure on $\mathbb{R}_+$ such that $|\mu|(K) < \infty$ for all compact sets $K \subset \mathbb{R}_+$, where $|\mu|$ denotes the total variation of μ. Denoting the Lebesgue measure on $\mathbb{R}$ by μ_L, let $w \colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a locally $(\mu_L \otimes \mu)$-integrable function and let $w_0 \in \mathbb{R}$. The operator $\mathcal{P}_\zeta \colon C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R})$ defined by

$$(\mathcal{P}_\zeta u)(t) = \int_0^\infty \int_0^{(\mathcal{B}_{r,\zeta(r)}(u))(t)} w(s,h)\mu_L(ds)\mu(dr) + w_0 \quad \forall u \in C(\mathbb{R}_+, \mathbb{R}) \quad \forall t \in \mathbb{R}_+, \tag{2.8}$$

is called a *Preisach* operator (see [4]). Clearly, $\mathcal{P}_\zeta$ is a hysteresis operator and, under the assumption that the measure μ is finite and w is essentially bounded, the operator $\mathcal{P}_\zeta$ is Lipschitz continuous with Lipschitz constant $|\mu|(\mathbb{R}_+)\|w\|_\infty$, see [40]. Furthermore, if the additional assumption that μ and w are non-negative valued is imposed, then, as shown in [40], the operator (2.8) is of class $\mathcal{T}_0^1$.

Setti $(\mathcal{P}_\zeta u)(t) = \int_0^\infty (\mathcal{B}_{r,\zeta(r)}(u))(t)\mu(dr)$ ye obtain the Prandtl operator $\mathcal{P}_\zeta \colon C(\mathbb{R}_+, \mathbb{R}) \to$ $C(\mathbb{R}_+,$

$$(\mathcal{P}_\zeta u)(t) = \int_0^\infty (\mathcal{B}_{r,\zeta(r)}(u))(t)\mu(dr) \quad \forall u \in C(\mathbb{R}_+, \mathbb{R}) \quad \forall t \in \mathbb{R}_+. \tag{2.9}$$

We illustrate the Prandtl operator for the case in which $\zeta = 0$ and the measure μ is given by $\mu(E) = \mu_L(E \cap [0,5])$. The operator takes the form

$$(\mathcal{P}_0 u)(t) = \int_0^5 (\mathcal{B}_{r,0}(u))(t)dr \quad \forall u \in C(\mathbb{R}_+, \mathbb{R}) \quad \forall t \in \mathbb{R}_+$$

and is illustrated in Figures 2-4 and 2-5.

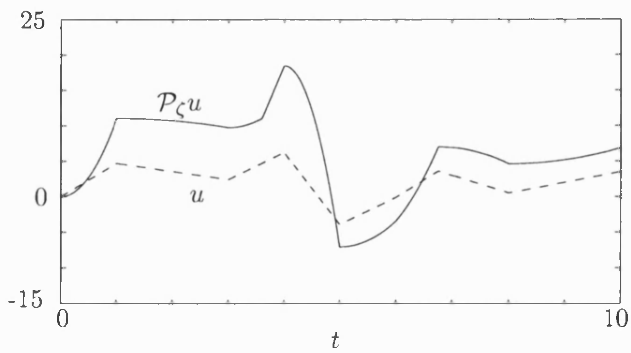


Figure 2-4: Prandtl hysteresis.

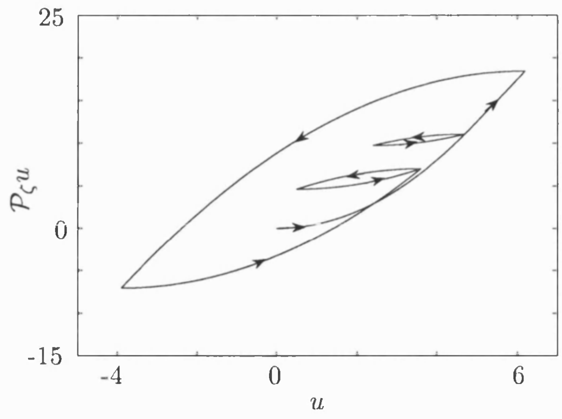


Figure 2-5: Input-output behaviour of Prandtl hysteresis.

Chapter 3

Existence theory for functional differential equations and inclusions

The operator class $\mathcal{T}_h^m$, introduced in Chapter 2, facilitates the investigation of large classes of nonlinear systems in later chapters. These systems take the form of (controlled) functional differential equations or inclusions. The focus of this chapter will be the development of existence theories for both functional differential equations and inclusions.

3.1 Functional differential equations

Let $h \geq 0$ and T be a causal operator of class $\mathcal{T}_h^m$. Let $\mathcal{D}$ be a domain in $\mathbb{R}_+ \times \mathbb{R}^m$, that is, a non-empty, connected, relatively open subset of $\mathbb{R}_+ \times \mathbb{R}^n$. Define $\mathcal{G} := \mathcal{D} \times \mathbb{R}^q$ and let $f \colon \mathcal{G} \to \mathbb{R}^m$ be a Carathéodory function. For $t_0 \geq 0$ consider the initial-value problem

$$\left.\begin{aligned} &\dot{y}(t) = f(t, y(t), (Ty)(t)), && (t, y(t)) \in \mathcal{D}, \\ &y|_{[-h,t_0]} = y^0 \in C([-h, t_0], \mathbb{R}^m), && (t_0, y^0(t_0)) \in \mathcal{D}. \end{aligned}\right\} \tag{3.1}$$

By a solution of (3.1), we mean a function $y \in C(I, \mathbb{R}^m)$ for some interval I of the form $[-h, \rho]$, $t_0 < \rho < \infty$ or $[-h, \omega)$, $t_0 < \omega \leq \infty$, such that $y|_{[-h,t_0]} = y^0$, $y|_J$ is locally absolutely continuous, $\dot{y}(t) = f(t, y(t), (Ty)(t))$ for almost all $t \in J$ and $(t, y(t)) \in \mathcal{D}$ for all $t \in J$, where $J := I \setminus [-h, t_0)$. A solution is said to be maximal if, and only if, it has no proper right extension that is also a solution. A solution defined on $[-h, \infty)$ is said to be *global*. An existence result for an initial-value problem similar to (3.1), in which the domain $\mathcal{D}$ is simply the space $\mathbb{R}_+ \times \mathbb{R}^n$ and t_0 takes the specific value 0, was provided in [25]. Both proofs are inspired by the existence result in [10, Chapter 2].

Theorem 3.1.1 *For each $t_0 \geq 0$ and $y^0 \in C([-h,t_0],\mathbb{R}^m)$ with $(t_0, y^0(t_0)) \in \mathcal{D}$,*

(i) the initial-value problem (3.1) *has a solution,*

(ii) every solution of (3.1) *can be extended to a maximal solution $y\colon [-h,\omega) \to \mathbb{R}^m$,*

(iii) if $y\colon [-h,\omega) \to \mathbb{R}^m$ is a maximal solution of (3.1) *and $\omega < \infty$, then, for every $\sigma \in [t_0,\omega)$ and compact set $\mathcal{K} \subset \mathcal{D}$, there exists $t \in [\sigma,\omega)$ with the property that $(t,y(t)) \notin \mathcal{K}$.*

Proof. **(i)** By Property (iii) of Definition 2.1.1 and since $\mathcal{D}$ is relatively open, there exist $\tau > t_0$, $c_0 > 0$ and $\delta > 0$ such that, for all $y,z \in \mathcal{C}(y^0;h,t_0,\tau,\delta)$, the following holds

$$\operatorname*{ess\text{-}sup}_{t\in[t_0,\tau]} \|(Ty)(t) - (Tz)(t)\| \leq c_0 \sup_{t\in[t_0,\tau]} \|y(t) - z(t)\|\,.$$

We may assume, without loss of generality, that $\delta \in (0,1)$ and $\tau > t_0$ are sufficiently small so that $[t_0,\tau] \times \overline{\mathbb{B}}_\delta(y^0(t_0)) \subset \mathcal{D}$. Evidently

$$\|y(t)\| < \max_{s\in[-h,t_0]} \|y^0(s)\| + \delta =: c_1 \quad \forall\, t \in [-h,\tau)$$

and so, by Property (iv) of the operator T, there exists $c_2 > 0$ such that

$$\|(Ty)(t)\| < c_2 \quad \text{for a.a.} \quad t \in [t_0,\tau].$$

Since f is a Carathéodory function, there exists an integrable function $\kappa\colon [t_0,\tau] \to \mathbb{R}$ such that

$$\|f(t,y,w)\| \leq \kappa(t) \quad \forall\, (t,y,w) \in [t_0,\tau] \times \overline{\mathbb{B}}_\delta(y^0(t_0)) \times \overline{\mathbb{B}}_{c_2}(0). \tag{3.2}$$

Define $\Gamma\colon [-h,\tau] \to \mathbb{R}_+$ by

$$\Gamma(t) := \begin{cases} 0, & t \in [-h,t_0), \\ \int_{t_0}^t \kappa(s)ds, & t \in [t_0,\tau] \end{cases}$$

and define $\rho := t_0 + \beta$, where $0 < \beta < \tau - t_0$, such that $\Gamma(\rho) < \delta$.

The next step is to construct a sequence (y_n) of continuous functions $[-h,\rho] \to \mathbb{R}_+$, with $(t,y_n(t)) \in \mathcal{D}$ for all $t \in [-h,\rho]$, as follows. Let $n \in \mathbb{N}$. For $i = 1,...,n$, define the sequence $\rho_n^i := t_0 + i\beta/n$ and functions $y_n^i \colon [-h, \rho_n^i] \to \mathbb{R}^m$ by the recursive formula:

$$i = 1: \quad y_n^1(t) := \begin{cases} y^0(t), & t \in [-h, t_0], \\ y^0(t_0), & t \in (t_0, \rho_n^1], \end{cases}$$

$$i > 1: \quad y_n^i(t) := \begin{cases} y_n^{i-1}(t), & t \in [-h, \rho_n^{i-1}], \\ y^0(t_0) + \int_{t_0}^{t-\beta/n} f(s, y_n^{i-1}(s), (Ty_n^{i-1})(s))ds, & t \in (\rho_n^{i-1}, \rho_n^i]. \end{cases}$$

Observe that, if $i \in \{1, ..., n-1\}$ and $\|y_n^i(t)\| < c_1$ for all $t \in [-h, \rho_n^i]$, then we have (a) $\|y_n^{i+1}(t)\| < c_1$ for all $t \in [-h, \rho_n^i]$ and (b) $\|(Ty_n^i)(t)\| < c_2$ for all $t \in [t_0, \rho_n^i]$ which, in turn, implies that, for all $t \in (\rho_n^i, \rho_n^{i+1}]$,

$$\begin{aligned} \|y_n^{i+1}(t) - y^0(t_0)\| &\le \int_{t_0}^{t-\beta/n} \|f(s, y_n^i(s), (Ty_n^i)(s))\| ds \\ &\le \int_{t_0}^{t-\beta/n} \kappa(s) ds \\ &= \Gamma(t - \beta/n) < \delta. \end{aligned}$$

Noting that $\|y_n^1(t)\| \le \max_{t \in [-h, t_0]} \|y^0(t)\| < c_1$ for all $t \in [-h, \rho_n^1]$, we may now infer (by induction on i) that

$$\|y_n^i(t)\| < c_1 \quad \forall\, i \in \{1, ..., n\} \quad \forall\, t \in [-h, \rho_n^i].$$

For notational convenience, we write $y_n := y_n^n$. By causality of T, the sequence (y_n), so constructed, has the property that, for each $n \in \mathbb{N}$,

$$y_n(t) = \begin{cases} y^0(t), & t \in [-h, t_0], \\ y^0(t_0), & t \in (t_0, \rho_n^1], \\ y^0(t_0) + \int_{t_0}^{t-\beta/n} f(s, y_n(s), (Ty_n)(s))ds, & t \in (\rho_n^1, \rho]. \end{cases} \tag{3.3}$$

Moreover, for all $n \in \mathbb{N}$, $\|y_n(t)\| < c_1$ for all $t \in [-h, \rho]$ and so the sequence (y_n) is uniformly bounded.

We next prove that the sequence (y_n) is equicontinuous. Let $\epsilon > 0$. On the closed interval $[t_0, \rho]$, Γ is uniformly continuous and so there exists some $\delta^* > 0$ such that, for $s, t \in [t_0, \rho]$,

$$|t - s| < \delta^* \implies |\Gamma(t) - \Gamma(s)| < \epsilon.$$

Let $n \in \mathbb{N}$, $s, t \in [t_0, \rho]$ with $|t - s| < \delta^*$. Without loss of generality, we assume that $s \leq t$. We consider three cases.

First, if $t_0 \leq s \leq t \leq \rho_n^1$, then $\|y_n(t) - y_n(s)\| = 0$. Secondly, if $t_0 < s \leq \rho_n^1 \leq t \leq \rho$, then $t - \rho_n^1 < \delta^*$ and so

$$\|y_n(t) - y_n(s)\| = \|y_n(t) - y^0(t_0)\| \leq \Gamma(t - \beta/n) < \epsilon.$$

Thirdly, if $\rho_1^n \leq s \leq t \leq \rho$ then

$$\|y_n(t) - y_n(s)\| \leq |\Gamma(t - \rho/n) - \Gamma(s - \rho/n)| < \epsilon.$$

Recalling that $y_n|_{[-h,t_0]} = y^0$ for all n, we conclude that the sequence (y_n) is equicontinuous. Since (y_n) is such that $(t, y_n(t)) \in [t_0, \tau] \times \overline{\mathbb{B}}_\delta(y^0(t_0)) \subset \mathcal{D}$ for all $n \in \mathbb{N}$ and all $t \in [t_0, \rho]$, we may apply the Arzelà-Ascoli theorem (see Appendix A, Theorem A.2.1), extracting a subsequence if necessary, to conclude that the sequence (y_n) converges uniformly on $[-h, \rho]$ to a continuous limit which we denote by y. Clearly $y|_{[-h,t_0]} = y^0$ and $(t, y(t)) \in \mathcal{D}$ for all $t \in [t_0, \rho]$.

By Property (iii) of the operator T, $\lim_{n\to\infty}(Ty_n)(t) = (Ty)(t)$ for almost all $t \in [t_0, \rho]$ and so, by the continuity of the function $f(t, \cdot, \cdot)$,

$$\lim_{n\to\infty} f(t, y_n(t), (Ty_n)(t)) = f(t, y(t), (Ty)(t)) \quad \text{for a.a. } t \in [t_0, \rho].$$

Noting that $\|(Ty_n)(s)\| < c_2$ for all $s \in [t_0, \rho]$ and invoking (3.2), we have

$$\|f(s, y(s), (Ty)(s))\| \leq \kappa(s) \quad \forall s \in [t_0, \rho] \quad \forall n \in \mathbb{N}.$$

Therefore,

$$\lim_{n\to\infty} \int_{t-\beta/n}^{t} f(s, y_n(s), (Ty_n)(s))ds = 0 \quad \forall\, t \in (t_0, \rho] \tag{3.4}$$

and, by the Lebesgue dominated convergence theorem (see Appendix A.2),

$$\lim_{n\to\infty} \int_{t_0}^{t} f(s, y_n(s), (Ty_n)(s))ds = \int_{t_0}^{t} f(s, y(s), (Ty)(s))ds \quad \forall\, t \in [t_0, \rho]. \tag{3.5}$$

By (3.3), (3.4) and (3.5), it follows that

$$y(t) = \begin{cases} y^0(t), & t \in [-h, t_0], \\ y^0(t_0) + \int_{t_0}^{t} f(s, y(s), (Ty)(s))ds, & t \in (t_0, \rho]. \end{cases}$$

Therefore, y is a solution of the initial-value problem (3.1).

(ii) Let $y \in C(I_y, \mathbb{R}^m)$ be a solution of (3.1). Define

$$\mathcal{A} := \left\{(I, z) \mid I_y \subset I,\ z \in C(I, \mathbb{R}^m) \text{ is a solution of (3.1) with } z|_{I_y} = y\right\}.$$

On this non-empty set define a partial order $\preceq$ by

$$(I_1, z_1) \preceq (I_2, z_2) \quad \Longleftrightarrow \quad \sup I_1 \leq \sup I_2 \quad \text{and} \quad z_2|_{I_1} = z_1.$$

We proceed to show that $\mathcal{A}$ has a maximal element, that is, an element $(I^*, z^*) \in \mathcal{A}$ such that, for all $(I, z) \in \mathcal{A}$, $(I^*, z^*) \preceq (I, z)$ implies $(I, z) = (I^*, z^*)$, in which case $z^* \in C(I^*, \mathbb{R}^m)$ is a solution of (3.1) and is a maximal extension of the solution $y \in C(I_y, \mathbb{R}^m)$. Let $\mathcal{O}$ be a totally ordered subset of $\mathcal{A}$. Let $\omega := \sup\{\sup I | (I, z) \in \mathcal{O}\}$ and let $z^*\colon [-h, \omega) \to \mathbb{R}^m$ be defined by the property that, for every $(I, z) \in \mathcal{O}$, $z^*|_I = z$. Then (ω, z^*) is in $\mathcal{A}$ and is an upper bound for $\mathcal{O}$. By Zorn's Lemma (see Appendix A, Lemma A.2.4), it follows that $\mathcal{A}$ contains at least one maximal element.

(iii) Assume $y \in C([-h, \omega)\mathbb{R}^m)$ is a maximal solution of (3.1) and that $\omega < \infty$. Seeking a contradiction, suppose there exist $\sigma \in [t_0, \omega)$ and compact $\mathcal{K} \subset \mathcal{D}$ such that $(t, y(t)) \in \mathcal{K}$ for all $t \in [\sigma, \omega)$. By boundedness of y and property (iv) of $\mathcal{T}_h$, we conclude that Ty is essentially bounded. Therefore, the function $t \mapsto (t, y(t), (Ty)(t))$ is essentially bounded and so, by continuity of f, it follows that $\dot{y}$ is essentially bounded on the interval $[t_0, \omega)$. Therefore y is uniformly continuous on $[-h, \omega)$ and so extends to a function $y^* \in C([-h, \omega], \mathbb{R}^m)$. Compactness of $\mathcal{K}$ implies that we have $(\omega, y^*(\omega)) \in \mathcal{K} \subset \mathcal{D}$. An application of the result of part (i), in the context of an initial-value problem of the form (3.1), with ω replacing t_0 and y^* replacing y^0, yields the existence of a solution $y^e \in C([-h\,\rho], \mathbb{R}^m)$ for some $\rho > \omega$, with $y^e|_{[-h,\omega)} = y$, contradicting maximality of y. $\square$

Imposing a local Lipschitz assumption, uniqueness of solutions to functional differential equations can also be established. A function $g\colon \mathcal{G} \to \mathbb{R}^m$ is said to be locally Lipschitz in its second and third arguments, locally uniformly with respect to its first argument, if

$$\left.\begin{array}{l} \forall \text{ compact } \Lambda \subset \mathcal{G}\ \exists L > 0 \text{ s.t.} \\ \|g(t,x,u) - g(t,y,v)\| \leq L\Big[\|x-y\| + \|u-v\|\Big] \quad \forall (t,x,u), (t,y,v) \in \Lambda. \end{array}\right\} \tag{3.6}$$

The following result is incorporated for independent interest, but will not be explicitly called on during this thesis and so the proof (which relies on a fixed point argument akin to that of [41]) is relegated to the appendix (see Appendix B.1).

Lemma 3.1.2 *Let* $g\colon \mathcal{G} \to \mathbb{R}^m$ *satisfy* (3.6). *For each* $t_0 \geq 0$ *and* $y^0 \in C([-h, t_0], \mathbb{R}^m)$ *with* $(t_0, y^0(t_0)) \in \mathcal{D}$, *the initial-value problem*

$$\dot{y}(t) = g(t, y(t), (Ty)(t)), \quad y|_{[-h,t_0]} = y^0 \in C([-h, t_0], \mathbb{R}^m), \quad (t_0, y^0(t_0)) \in \mathcal{D}, \quad (3.7)$$

has a unique maximal solution, $y\colon [-h, \omega) \to \mathbb{R}^m$. *Furthermore, if* $\omega < \infty$, *then for every* $\sigma \in [t_0, \omega)$ *and compact set* $\mathcal{K} \subset \mathcal{D}$ *there exists* $t \in [\sigma, \omega)$ *such that* $(t, y(t)) \notin \mathcal{K}$.

3.2 Existence theory for functional differential inclusions

In Chapter 6, a potentially discontinuous controller will be applied in a tracking problem in the context of a class of nonlinear systems modelled by functional differential equations. The potential discontinuity will be embedded in a set-valued map and interpreted accordingly. The set-valued nature of the resulting closed-loop system necessitates the development of a suitable existence theory and this is the main focus of the current section.

The area of set-valued analysis is well developed, see the background results in Appendix C and references therein. Results pertaining to the existence of solutions for differential inclusions can be found in [2] and [15], for example, though functional differential inclusions are not considered. The existence theory in the present text is influenced by [9, Theorem 3.1.7] and [39, Theorem 2D.5] though it essentially builds on the result proved in [59, Chapter 3].

Let $\mathcal{D}$ be a domain in $\mathbb{R}_+ \times \mathbb{R}^m$. Let $(t, y, w) \mapsto G(t, y, w) \subset \mathbb{R}^m$ be upper semicontinuous on $\mathcal{G} := \mathcal{D} \times \mathbb{R}^q$, with non-empty, convex and compact values. Let $h \geq 0$ and T be a causal operator of class $\mathcal{T}_h^m$. For $t_0 \geq 0$, consider the initial-value problem

$$\dot{y}(t) \in G(t, y(t), (Ty)(t)), \quad y|_{[-h,t_0]} = y^0 \in C([-h, t_0], \mathbb{R}^m), \quad (t_0, y^0(t_0)) \in \mathcal{D}. \quad (3.8)$$

By a *solution* of (3.8), we mean a function $y \in C(I, \mathbb{R}^m)$ for some interval I of the form $[-h, \rho]$, $t_0 < \rho < \infty$ or $[-h, \omega)$, $t_0 < \omega \leq \infty$, such that $y|_{[-h,t_0]} = y^0$, $y|_J$ is locally absolutely continuous, $\dot{y}(t) \in G(t, y(t), (Ty)(t))$ for almost all $t \in J$, and $(t, y(t)) \in \mathcal{D}$ for all $t \in J$, where $J := I \setminus [-h, t_0)$. A solution is said to be *maximal* if it has no proper right extension that is also a solution. The main goal of this chapter is to establish the following.

Theorem 3.2.1 *For each* $t_0 \geq 0$ *and* $y^0 \in C([-h, t_0], \mathbb{R}^m)$ *with* $(t_0, y^0(t_0)) \in \mathcal{D}$,

(i) *the initial-value problem* (3.8) *has a solution,*

(ii) *every solution can be extended to a maximal solution* $y\colon [-h,\omega) \to \mathbb{R}^m$*,*

(iii) *if* $y\colon [-h,\omega) \to \mathbb{R}^m$ *is a maximal solution of* (3.8) *and* $\omega < \infty$*, then for every* $\sigma \in [t_0,\omega)$ *and every compact set* $\mathcal{K} \subset \mathcal{D}$*, there exists* $t \in [\sigma,\omega)$ *such that* $(t, y(t)) \notin \mathcal{K}$.

Proof. **(i)** Let $(\varepsilon_n) \subset (0,1)$ be a monotonically decreasing sequence with $\varepsilon_n \to 0$ as $n \to \infty$. By the Approximate Selection Theorem (see Appendix C, Theorem C.1.8), for each $n \in \mathbb{N}$, there exists a locally Lipschitz function $g_n\colon \mathcal{G} \to \mathbb{R}^m$ with

$$\operatorname{graph}(g_n) \subset \operatorname{graph}(G) + \mathbb{B}_{\varepsilon_n}. \tag{3.9}$$

By Theorem 3.1.1, for each $n \in \mathbb{N}$, the initial-value problem

$$\dot{y}(t) = g_n(t, y(t), (Ty)(t)), \quad y|_{[-h,t_0]} = y^0 \in C([-h,t_0],\mathbb{R}^m), \quad (t_0, y^0(t_0)) \in \mathcal{D},$$

has a maximal solution which we denote by $y_n\colon [-h,\omega_n) \to \mathbb{R}^m$.

Recalling that $\mathcal{D}$ is a relatively open subset of $\mathbb{R}_+ \times \mathbb{R}^m$ and invoking property (iii) of $T \in \mathcal{T}_h^m$, we may choose $\delta > 0$ sufficiently small and $\omega^* > t_0$ sufficiently close to t_0 so that

$$[t_0,\omega^*] \times \overline{\mathbb{B}}_\delta(y^0(t_0)) =: \mathcal{K}_0 \subset \mathcal{D}$$

and there exists $c_0 > 0$ such that

$$\operatorname*{ess\text{-}sup}_{t\in[t_0,\omega^*]} \|(Ty)(t)-(Tz)(t)\| \le c_0 \max_{t\in[t_0,\omega^*]} \|y(t)-z(t)\| \quad \forall\, y,z \in \mathcal{C}(y^0; h, t_0, \omega^*, \delta). \tag{3.10}$$

For each $n \in \mathbb{N}$, define

$$\omega_n^* := \min\{\omega^*, \omega_n\}, \qquad \Delta_n := \{t \in [t_0,\omega_n^*) \mid \|y_n(t) - y^0(t_0)\| = \delta\}$$

and

$$\rho_n := \begin{cases} \inf \Delta_n, & \text{if } \Delta_n \ne \emptyset, \\ \omega_n^*, & \text{if } \Delta_n = \emptyset. \end{cases}$$

We claim that $\rho_n < \omega_n$ for all $n \in \mathbb{N}$. Suppose otherwise, then there exists $n \in \mathbb{N}$ such that $\rho_n = \omega_n$. It follows that $\Delta_n = \emptyset$ and so $\omega_n = \omega_n^* \le \omega^*$. Therefore, $(t, y_n(t)) \in \mathcal{K}_0 \subset \mathcal{D}$ for all $t \in [t_0,\omega_n)$, contradicting the final assertion of Lemma 3.1.2. Therefore, $\rho_n < \omega_n$ for all $n \in \mathbb{N}$. Furthermore, for each $n \in \mathbb{N}$, $y_n(t) \in \overline{\mathbb{B}}_\delta(y^0(t_0))$ for all $t \in [t_0, \rho_n]$ and so

$$\|y_n(t)\| \leq c_1 := \max_{s\in[-h,t_0]} \|y^0(s)\| + \delta \quad \text{for all } t \in [-h, \rho_n] \text{ and all } n \in \mathbb{N}.$$

By property (iv) of $T \in \mathcal{T}_h^m$, there exists $c_2 > 0$ such that

$$\|(Ty_n)(t)\| \leq c_2 \quad \text{for a.a. } t \in [t_0, \rho_n] \text{ and all } n \in \mathbb{N}.$$

Write $\mathcal{K}_1 := \mathcal{K}_0 \times \overline{\mathbb{B}}_{c_2}$ and observe

$$(t, y_n(t), (Ty_n)(t)) \in \mathcal{K}_1 \quad \text{for a.a. } t \in [t_0, \rho_n] \text{ and all } n \in \mathbb{N}.$$

Since G is upper semicontinuous with compact values, we may apply Proposition C.1.7 (see Appendix C) to conclude that the set $\mathcal{K}_2 := G(\mathcal{K}_1)$ is compact. Let $c_3 := 1 + \max_{v\in\mathcal{K}_2} \|v\|$. Then, in view of (3.9),

$$\|g_n(t, y, w)\| < c_3 \quad \text{for all } (t, y, w) \in \mathcal{K}_1 \text{ and all } n \in \mathbb{N}. \tag{3.11}$$

Therefore,

$$\|y_n(\rho_n) - y^0(t_0)\| \leq \int_{t_0}^{\rho_n} \|\dot{y}_n(t)\|dt = \int_{t_0}^{\rho_n} \|g_n(t, y_n(t), (Ty_n)(t))\|dt$$
$$< c_3|\rho_n - t_0| \quad \forall\, n \in \mathbb{N}. \tag{3.12}$$

Next, define $\rho := \inf_{n\in\mathbb{N}} \rho_n \geq t_0$. Seeking a contradiction, suppose $\rho = t_0$. Fix $n \in \mathbb{N}$ sufficiently large so that $c_3|\rho_n - t_0| < \delta$ and $\rho_n < \omega^*$. Recalling that $\rho_n < \omega_n$, we have $\rho_n < \min\{\omega^*, \omega_n\} = \omega_n^*$ and so $\Delta_n \neq \emptyset$ and we arrive at a contradiction:

$$\delta = \|y_n(\rho_n) - y^0(t_0)\| < c_3|\rho_n - t_0| < \delta.$$

Therefore $\rho \in (t_0, \omega^*]$. For each $n \in \mathbb{N}$, define

$$z_n := y_n|_{[t_0,\rho]} \quad \text{and} \quad w_n := (Ty_n)|_{[t_0,\rho]}.$$

For all $t \in [t_0, \rho]$, $(z_n(t)) \subset \overline{\mathbb{B}}_\delta(y^0(t_0))$ and by (3.11),

$$\|\dot{z}_n(t)\| < c_3 \quad \text{for a.a. } t \in [t_0, \rho] \text{ and all } n \in \mathbb{N}. \tag{3.13}$$

Therefore, the sequence $(z_n) \subset C([t_0, \rho], \mathbb{R}^m)$ is uniformly bounded and equicontinuous. By the Arzelà-Ascoli theorem, and extracting a subsequence if necessary, we may assume that (z_n) converges uniformly to $z \in C([t_0, \rho], \mathbb{R}^m)$.

To complete the proof, we adopt an argument akin to those adopted in the proof of [9, Theorem 3.1.7] and [39, Theorem 2D.5].

By weak*-compactness of the unit ball in $L^\infty([t_0, \rho], \mathbb{R}^m)$ (Alaoglu's theorem, see Appendix A.2), together with (3.13), the sequence $(\dot{z}_n) \subset L^\infty([t_0, \rho], \mathbb{R}^m) \subset L^1([t_0, \rho], \mathbb{R}^m)$ has a subsequence (which we do not relabel) with weak*-limit $v \in L^\infty([t_0, \rho], \mathbb{R}^m)$, that is,

$$\lim_{n\to\infty} \int_{t_0}^{\rho} \langle p(t), \dot{z}_n(t)\rangle \, dt = \int_{t_0}^{\rho} \langle p(t), v(t)\rangle \, dt \quad \forall\, p \in L^1([t_0, \rho], \mathbb{R}^m) \tag{3.14}$$

and so, a fortiori, the sequence $(\dot{z}_n)$ converges weakly in $L^1([t_0, \rho], \mathbb{R}^m)$ to v. Let $\{e_1, \cdots, e_m\}$ be a basis for $\mathbb{R}^m$. For $k = 1, ..., m$ and $t \in [t_0, \rho]$, define $p_{k,t} \in L^1([t_0, \rho], \mathbb{R}^m)$ by

$$p_{k,t}(s) := \begin{cases} e_k, & s \in [t_0, t], \\ 0, & \text{otherwise.} \end{cases}$$

Setting $p = p_{k,t}$ ($k = 1, ..., m$ and $t \in [t_0, \rho]$) in (3.14) and integrating, we may now conclude that

$$z(t) = \lim_{n\to\infty} z_n(t) = y^0(t_0) + \int_{t_0}^{t} v(s) ds \quad \forall\, t \in [t_0, \rho].$$

Therefore, $z \in AC([t_0, \rho], \mathbb{R}^m)$ and $\dot{z}(t) = v(t)$ for almost all $t \in [t_0, \rho]$.

Let $y \in C([-h, \rho], \mathbb{R}^m)$ denote the concatenation of y^0 and z, and write $w := (Ty)|_{[t_0,\rho]}$. Therefore, $y|_{[-h,t_0]} = y^0$, $y|_{[t_0,\rho]} = z \in AC([t_0, \rho], \mathbb{R}^m)$ and, to conclude that y is a solution of the initial-value problem (3.8), it suffices to show that $\dot{z}(t) \in G(t, z(t), w(t))$ for almost all $t \in [t_0, \rho]$.

By (3.10), we have

$$\|w_n(t) - w(t)\| \le c_0 \max_{s\in[t_0,\rho]} \|z_n(s) - z(s)\| \quad \text{for a.a. } t \in [t_0, \rho] \text{ and all } n \in \mathbb{N}. \tag{3.15}$$

Therefore, for almost all $t \in [t_0, \rho]$, $w_n(t) \to w(t)$ as $n \to \infty$. Moreover,

$$\int_{t_0}^{\rho} \|w_n(t) - w(t)\| dt \le c_0 |\rho - t_0| \max_{s\in[t_0,\rho]} \|z_n(s) - z(s)\| \to 0 \quad \text{as } n \to \infty.$$

Therefore, (w_n) converges (strongly) in $L^1([t_0, \rho], \mathbb{R}^m)$ to w.

Define the function $\sigma\colon \mathcal{K}_1 \times \mathbb{R}^m \to \mathbb{R}$ by

$$\sigma(t,\eta,\xi,q) := \max\{\langle q,\zeta\rangle \mid \zeta \in G(t,\eta,\xi)\}.$$

Observe that, for each $(t,\eta,\xi) \in \mathcal{K}_1$, $q \mapsto \sigma(t,\eta,\xi,q)$ is the support function (see Appendix C.2) of the compact, convex set $G(t,\eta,\xi)$ (and so is globally Lipschitz, see Theorem C.2.2(ii)). Therefore, to establish that $\dot{z}(t) \in G(t,z(t),w(t))$ for almost all $t \in [t_0,\rho]$, it follows from Theorem C.2.2(i) (see Appendix C.2) that it suffices to show that

$$\langle q,\dot{z}(t)\rangle \le \sigma(t,z(t),w(t),q) \quad \text{for a.a. } t \in [t_0,\rho] \text{ and all } q \in \mathbb{R}^m. \tag{3.16}$$

By continuity of the maps $q \mapsto \langle q,\zeta\rangle$ and $q \mapsto \sigma(t,\eta,\xi,q)$ for all $\zeta \in \mathbb{R}^m$ and all $(t,\eta,\xi) \in \mathcal{K}_1$, (3.16) holds if, any only if,

$$\langle q,\dot{z}(t)\rangle \le \sigma(t,z(t),w(t),q) \quad \text{for a.a. } t \in [t_0,\rho] \text{ and all } q \in \mathbb{Q}^m, \tag{3.17}$$

where $\mathbb{Q}^m \subset \mathbb{R}^m$ is the set of vectors in $\mathbb{R}^m$ with rational coordinates. We proceed to establish (3.17). First, we show that, for each $q \in \mathbb{R}^m$, the map $(t,\eta,\xi) \mapsto \sigma(t,\eta,\xi,q)$ is upper semicontinuous on $\mathcal{G}$. Let $q \in \mathbb{R}^m$ and $(t,\eta,\xi) \in \mathcal{K}_1$ be arbitrary and define

$$\sigma^* := \limsup_{(t',\eta',\xi')\to(t,\eta,\xi)} \sigma(t',\eta',\xi',q).$$

Let $\big((t_k,\eta_k,\xi_k)\big) \subset \mathcal{K}_1$ be a sequence converging to (t,η,ξ) such that $\sigma(t_k,\eta_k,\xi_k,q) \to \sigma^*$ as $k \to \infty$. For each $k \in \mathbb{N}$, by compactness of $G(t_k,\eta_k,\xi_k)$ there exists $\zeta_k \in G(t_k,\eta_k,\xi_k)$ such that $\langle q,\zeta_k\rangle = \sigma(t_k,\eta_k,\xi_k,q)$. The resulting sequence (ζ_k) is contained in the compact set $\mathcal{K}_2 = G(\mathcal{K}_1)$ and so has a subsequence converging to $\zeta \in \mathcal{K}_2$. By Proposition C.1.6 (see Appendix C), the graph of G is closed and so we may infer that $\zeta \in G(t,\eta,\xi)$. Therefore,

$$\begin{aligned}\limsup_{(t',\eta',\xi')\to(t,\eta,\xi)} \sigma(t',\eta',\xi',q) &= \lim_{k\to\infty} \sigma(t_k,\eta_k,\xi_k,q)\\ &= \lim_{k\to\infty} \langle q,\zeta_k\rangle = \langle q,\zeta\rangle \le \sigma(t,\eta,\xi,q),\end{aligned}$$

whence upper semicontinuity of $\sigma(\cdot,\cdot,\cdot,q)$.

For $p \in L^\infty([t_0,\rho],\mathbb{R}^m)$,

$$|\sigma(t,z_n(t),w_n(t),p(t))| \le \max_{v\in\mathcal{K}_2} \|v\|\|p(t)\| \le c_3\|p(t)\| \quad \text{for a.a. } t \in [t_0,\rho] \text{ and all } n \in \mathbb{N}.$$

Furthermore, in view of (3.9),

$$\begin{aligned}\langle p(t), \dot{z}_n(t)\rangle &= \langle p(t), g_n(t, z_n(t), w_n(t))\rangle \\ &\le \sigma(t, z_n(t), w_n(t), p(t)) + \varepsilon_n \|p(t)\| \quad \text{for a.a. } t \in [t_0, \rho] \text{ and all } n \in \mathbb{N},\end{aligned}$$

and so

$$\int_{t_0}^{\rho} \left[\langle p(t), \dot{z}_n(t)\rangle - \varepsilon_n \|p(t)\|\right] \le \int_{t_0}^{\rho} \sigma(t, z_n(t), w_n(t), p(t))\, dt \quad \forall\, n \in \mathbb{N}.$$

Taking the limit superior as $n \to \infty$, invoking Fatou's lemma (see Appendix A.2) and upper semicontinuity of $\sigma(\cdot, \cdot, \cdot, q)$, we have

$$\int_{t_0}^{\rho} \langle p(t), \dot{z}(t)\rangle\, dt \le \int_{t_0}^{\rho} \sigma(t, z(t), w(t), p(t))\, dt. \tag{3.18}$$

Let $q \in \mathbb{Q}^m$ and let $t \in [t_0, \rho)$ be a Lebesgue point for the integrable functions $\dot{z}$ and $t \mapsto \sigma(t, z(t), w(t), q)$. For $\tau > 0$, define $p \in L^\infty([t_0, \rho], \mathbb{R}^m)$ by

$$p(s) := \begin{cases} q/\tau, & s \in [t, t+\tau] \cap [t_0, \rho], \\ 0, & \text{otherwise.} \end{cases}$$

By (3.18), we have

$$\frac{1}{\tau} \int_t^{t+\tau} \left[\sigma(s, z(s), w(s), q) - \langle q, \dot{z}(s)\rangle\right] ds \ge 0 \quad \forall\, \tau > 0.$$

Passage to the limit as $\tau \to 0$ yields $\langle q, \dot{z}(t)\rangle \le \sigma(t, z(t), w(t), q)$, which is valid for all $t \in [t_0, \rho] \setminus \mathcal{N}(q)$, where $\mathcal{N}(q)$ is a set of measure zero which may depend on $q \in \mathbb{Q}^m$. Since $\mathbb{Q}^m$ is countable, $\cup_{q \in \mathbb{Q}^m} \mathcal{N}(q)$ has measure zero and so we may conclude that (3.17) (and hence (3.16)) holds. We have shown that $y\colon [-h, \rho] \to \mathbb{R}^m$ is a solution of (3.8).

(ii) Let $y \in C(I_y, \mathbb{R}^m)$ be a solution of (3.8). Define

$$\mathcal{A} := \left\{(I, z) \mid I_y \subset I,\ z \in C(I, \mathbb{R}^m) \text{ is a solution of (3.8) with } z|_{I_y} = y\right\}.$$

On this non-empty set define a partial order $\preceq$ by

$$(I_1, z_1) \preceq (I_2, z_2) \quad \Longleftrightarrow \quad \sup I_1 \le \sup I_2 \quad \text{and} \quad z_2|_{I_1} = z_1.$$

We proceed to show that $\mathcal{A}$ has a maximal element, that is, an element $(I^*, z^*) \in \mathcal{A}$ such that, for all $(I, z) \in \mathcal{A}$, $(I^*, z^*) \preceq (I, z)$ implies $(I, z) = (I^*, z^*)$, in which case

$z^* \in C(I^*, \mathbb{R}^m)$ is a solution of (3.8) and is a maximal extension of the solution $y \in C(I_y, \mathbb{R}^m)$. Let $\mathcal{O}$ be a totally ordered subset of $\mathcal{A}$. Let $\omega := \sup\{\sup I | (I, z) \in \mathcal{O}\}$ and let $z^*\colon [-h, \omega) \to \mathbb{R}^m$ be defined by the property that, for every $(I, z) \in \mathcal{O}$, $z^*|_I = z$. Then (ω, z^*) is in $\mathcal{A}$ and is an upper bound for $\mathcal{O}$. By Zorn's Lemma, it follows that $\mathcal{A}$ contains at least one maximal element. This establishes assertion (ii).

(iii) Assume $y \in C([-h, \omega), \mathbb{R}^m)$ is a maximal solution of (3.8) and that $\omega < \infty$. Seeking a contradiction, suppose there exist $\sigma \in [t_0, \omega)$ and compact $\mathcal{K} \subset \mathcal{D}$ such that $(t, y(t)) \in \mathcal{K}$ for all $t \in [\sigma, \omega)$. By boundedness of y and property (iv) of $\mathcal{T}_h^m$, we conclude that Ty is bounded. Therefore, the function $t \mapsto (t, y(t), (Ty)(t))$ is essentially bounded and so by Proposition C.1.7 and properties of G, it follows that $\dot{y}$ is essentially bounded on $[\sigma, \omega)$. Therefore, y is uniformly continuous on $[-h, \omega)$ and so extends to a function $y^* \in C([-h, \omega], \mathbb{R}^m)$. By compactness of $\mathcal{K}$, we have $(\omega, y^*(\omega)) \in \mathcal{K} \subset \mathcal{D}$. An application of Assertion (i) of the theorem (with ω and y^* replacing t_0 and y^0, respectively) yields the existence of a solution $y^e \in C([-h, \rho], \mathbb{R}^m)$ for some $\rho > \omega$, with $y^e|_{[-h,\omega)} = y$. This contradicts maximality of y. □

Chapter 4

Approximate tracking for nonlinearly-perturbed linear systems with known relative degree

In this chapter, the approximate tracking and prescribed transient behaviour objectives introduced in Section 1.2 are considered for a class of nonlinearly-perturbed multi-input, multi-output, linear systems with known relative degree. The aim is to develop a control strategy ensuring that the reference signal is tracked by the system output y with prescribed asymptotic accuracy and guaranteed transient performance. This work constitutes an extension, to higher relative degree, of the approach adopted in [26] in which a performance funnel control objective is applied to a class of multi-input, multi-output systems in the restricted relative-degree-one case.

For the class considered in [26], the control law is a special case of (4.10) developed later in this chapter: the associated gain k is not monotone (non-decreasing) - which contrasts with typical high-gain adaptive control schemes; $k(t)$ becomes large only when the distance between the output and the funnel boundary becomes small which, in conjunction with the underlying high-gain properties of the system class, precludes boundary contact. Perhaps the most significant contribution of both the work in [26] and the present text is the consideration of the transient behaviour. The results in this chapter are based on [27].

4.1 Introduction

We begin by introducing the class of systems and follow this with some remarks on the related literature.

4.1.1 Class of systems

Definition 4.1.1 (System class $\mathcal{N}_\rho$) *For $\rho \in \mathbb{N}$, $\mathcal{N}_\rho$ is the class of nonlinearly-perturbed (perturbation p), m-input ($u(t) \in \mathbb{R}^m$), m-output ($y(t) \in \mathbb{R}^m$) systems (A, B, C, p) of the form*

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) + p(t, x(t)), \qquad x(0) = x^0 \in \mathbb{R}^n, \\ y(t) &= C\,x(t) \in \mathbb{R}^m, \end{aligned}\right\} \tag{4.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $p\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are such that the following assumptions hold.

(A1) *(known relative degree and sign-definite high-frequency gain)*
For some known $\rho \in \mathbb{N}$, $CA^iB = 0$ for $i = 1, ..., \rho - 2$ and $CA^{\rho-1}B$ is either positive or negative definite.

(A2) *(minimum-phase)*

$$\det \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \neq 0 \qquad \textit{for all } s \in \overline{\mathbb{C}}_+.$$

(A3) *(nonlinear perturbation)*
The perturbation $p\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function with the property that, for some continuous $\phi\colon \mathbb{R}^m \to \mathbb{R}_+$,

$$\|p(t, x)\| \leq \phi(Cx) \quad \forall\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Remark 4.1.2

(i) Recall from Section 1.4.6 that, if the transfer function $s \mapsto C(sI - A)^{-1}B = \sum_{i=0}^{\infty} CA^iBs^{-(i+1)}$ is non-trivial (not identically zero), then there exists $\rho \in \mathbb{N}$ such that $CA^iB = 0$ for $i = 1, ..., \rho - 2$ and $CA^{\rho-1}B \neq 0$. Assumption (A1) requires that ρ be known, and $CA^{\rho-1}B$ to be not only invertible, but also either positive or negative definite. Note that symmetry of $CA^{\rho-1}B$ is not a requirement.

In the single-input single-output case, the hypothesis of sign definiteness is redundant and (A1) is simply equivalent to positing that the transfer function is of known relative degree $\rho \geq 1$. In the multi-input, multi-output context, (A1) is restrictive: nevertheless the multi-input, multi-output case is included here as this can be done with little extra analytical effort *vis a vis* the single-input, single-output case. Linear systems satisfying Assumptions (A1) and (A2) are, at least in the single-input single-output case, typical of the class of systems underlying the area of high-gain adaptive control, as studied in [49], [8] and [45] for example.

(ii) Note that the minimum phase assumption implies that the unperturbed ($p \equiv 0$) system has exponentially stable zero dynamics, see, for example, [31, Section 5.1].

(iii) Even in the absence of a nonlinear perturbation p, the results in this chapter are new. Perturbations satisfying (A3) can be incorporated with relative ease in the analysis. A larger class of nonlinear systems, modelled by functional differential equations, will be considered in Chapter 5.

(iv) Paper [48], discussed in Section 1.4.7, considers the class of systems (4.1) with known relative degree, satisfying the minimum-phase assumption, restricted to the single-input, single-output case with high-frequency gain of known sign. The controller is adaptive with non-decreasing gain k, invokes a piecewise-constant switching strategy, and is less flexible in its scope for shaping transient behaviour (in particular, an *a priori* bound on the initial data is required).

4.1.2 Control objectives and the performance funnel

There are two control objectives, as follows:

(i) approximate tracking, by the output y, of reference signals r of class $\mathcal{R} := W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$. In particular, for arbitrary $\lambda > 0$, we seek an output feedback strategy which ensures that, for every $r \in \mathcal{R}$, the closed-loop system has bounded solution and the tracking error $e(t) = y(t) - r(t)$ is ultimately bounded by λ (that is, $\|e(t)\| \leq \lambda$ for all t sufficiently large);

(ii) prescribed transient behaviour of the tracking error signal.

We capture both objectives in the concept of a performance funnel, as introduced in Section 1.3,

$$\mathcal{F}_\varphi = \big\{(t,e) \in \mathbb{R}_+ \times \mathbb{R}^m \big| \ \ \varphi(t)\,\|e\| < 1\big\}$$

associated with a function φ (the reciprocal of which determines the funnel boundary) belonging to the space of functions Φ defined in Section 1.3, viz.

$$\Phi = \left\{ \varphi \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \;\middle|\; \varphi(0) = 0,\ \varphi(s) > 0 \ \forall\ s > 0,\ \liminf_{s \to \infty} \varphi(s) > 0 \right\}.$$

The aim is an output feedback strategy ensuring that, for every reference signal $r \in \mathcal{R}$, the tracking error $e = y - r$ evolves within the funnel $\mathcal{F}_\varphi$.

In the next section, Section 4.2, we describe the control strategy and present the closed-loop system. The main result, Theorem 4.4.1, follows in Section 4.4.

4.2 The control

Let assumptions (A1) and (A2) hold, with relative degree $\rho \geq 2$; the relative degree 1 case will be treated separately.

4.2.1 Filter

The control approach adopted in this chapter (and Chapter 5) invokes an input filter or linear pre-compensator of the form illustrated in Figure 4-1.

$$u \longrightarrow \boxed{(F, G)} \longrightarrow \xi$$

Figure 4-1: Filter/pre-compensator.

Fixing $\mu > 0$ (arbitrarily), we formally introduce the filter given by

$$\left.\begin{aligned} \dot{\xi}_i(t) &= -\mu \xi_i(t) + \xi_{i+1}, & \xi_i(0) &= \xi_i^0 \in \mathbb{R}^m, \quad i = 1, \ldots, \rho - 2, \\ \dot{\xi}_{\rho-1}(t) &= -\mu \xi_{\rho-1}(t) + u(t), & \xi_{\rho-1}(0) &= \xi_{\rho-1}^0 \in \mathbb{R}^m, \end{aligned}\right\} \tag{4.2}$$

which, on writing (wherein I and 0 denote the $m \times m$ identity and zero matrices)

$$\xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \\ \vdots \\ \xi_{\rho-2}(t) \\ \xi_{\rho-1}(t) \end{bmatrix}, \quad F = \begin{bmatrix} -\mu I & I & 0 & \cdots & 0 & 0 \\ 0 & -\mu I & I & \cdots & 0 & 0 \\ 0 & 0 & -\mu I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\mu I & I \\ 0 & 0 & 0 & \cdots & 0 & -\mu I \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I \end{bmatrix}, \tag{4.3}$$

may be expressed as

$$\dot{\xi}(t) = F\xi(t) + Gu(t), \qquad \xi(0) = \xi^0 \in \mathbb{R}^{(\rho-1)m}. \tag{4.4}$$

Remark 4.2.1

(i) For clarity of exposition, we have not chosen the most general presentation. The matrix F in the filter (4.4) could have arbitrary negative eigenvalues on the diagonal.

(ii) The intuition behind the filter (4.4) and the feedback control strategy, introduced in the next section, is as follows. Writing

$$H := \left[I \vdots 0 \vdots 0 \vdots \cdots \vdots 0 \vdots 0\right],$$

the transfer function from u to ξ_1, with $\xi^0 = 0$, is given by

$$H(sI - F)^{-1}G = (s + \mu)^{1-\rho} I\,.$$

Therefore, with reference to Figure 4-2 below, the transfer function from the signal ξ_1 to the output y is given by

$$(s+\mu)^{\rho-1} C(sI-A)^{-1}B = C[\mu I+A]^{\rho-1}(sI-A)^{-1}B = C(sI-A)^{-1}[\mu I+A]^{\rho-1}B.$$

As will be shown in Lemma 4.3.1, this transfer function has the minimum phase property and is of relative degree one: in other words, the triple $\left(A, [\mu I + A]^{\rho-1}B, C\right)$ defines a minimum-phase system of relative degree one with high-frequency gain $CA^{\rho-1}B$.

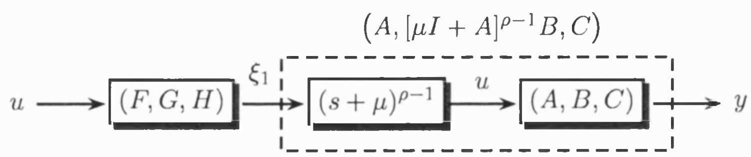


Figure 4-2: Input-output representation.

From existing results on relative degree one systems, see [26], and momentarily regarding ξ_1 as an independent input variable, it is known that, in the case wherein $CA^{\rho-1}B$ is positive definite, the choice $\xi_1 = -ke$, for an appropriately constructed gain k, achieves the control objectives for the system defined by the triple $(A, [\mu I + A]^{\rho-1}B, C)$; Theorem 4.4.2 extends this to the case of sign-definite $CA^{\rho-1}B$ of unknown sign, asserting that the choice $\xi_1 = -\gamma_1(k,e) = \nu(k)\,e$ (see Section 4.2.2) achieves the control objectives for the system. However, with $\rho \geq 2$, ξ_1 is not an independent input but instead is generated via the filter. The essence of the strategy is a procedure which "backsteps" through the filter variables to arrive at an input u which assures boundedness of the "mismatch" $\xi_1 - \nu(k)e$, which, in turn, ensures that the performance objectives are achieved (as will be shown in Theorem 4.4.1).

(iii) The approach used in this chapter is in the spirit of the adaptive results in [71] and the non-adaptive results in [26]. The paper [71] restricts the class of systems (4.1) satisfying Assumptions (A1) and (A2) to the single-input, single-output case; the control objective is (continuous) adaptive λ-tracking with non-decreasing gain; transient behaviour is not addressed, however nonlinear perturbations as in Assumption (A3) are allowed. The filter and the "backstepping" construction of the feedback strategy in this chapter is akin to that of [71] and the procedure also resembles the methodology of [33]. However, the controller in the latter incorporates a non-decreasing adaptive gain and achieves output stabilization.

4.2.2 Feedback

Let $\varphi \in \Phi$ and let $\nu \colon \mathbb{R} \to \mathbb{R}$ be any C^∞ function with the properties

$$\left.\begin{aligned} \limsup_{k\to\infty} \nu(k) &= +\infty, \\ \liminf_{k\to\infty} \nu(k) &= -\infty\,. \end{aligned}\right\} \tag{4.5}$$

Introduce the projections

$$\pi_i \colon \mathbb{R}^{(\rho-1)m} \to \mathbb{R}^{im}, \quad \xi = (\xi_1, \ldots, \xi_{\rho-1}) \mapsto (\xi_1, \ldots, \xi_i), \qquad i = 1, \ldots, \rho-1, \tag{4.6}$$

and define the C^∞ function

$$\gamma_1 \colon \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m, \qquad (k,e) \mapsto \gamma_1(k,e) := -\nu(k)e\,, \tag{4.7}$$

with derivative (Jacobian matrix function) $D\gamma_1$. Next, for $i = 2, \ldots, \rho$, define the C^∞ function $\gamma_i \colon \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{(i-1)m} \to \mathbb{R}^m$ by the recursion

$$\begin{aligned}\gamma_i(k, e, \pi_{i-1}\xi) &:= \gamma_{i-1}(k, e, \pi_{i-2}\xi) \\ &+ \|D\gamma_{i-1}(k, e, \pi_{i-2}\xi)\|^2 \, k^4 \, (1 + \|\pi_{i-1}\xi\|^2)\Big(\mu^{2-i}\xi_{i-1} + \gamma_{i-1}(k, e, \pi_{i-2}\xi)\Big), \end{aligned} \tag{4.8}$$

wherein we adopt the notational convention $\gamma_1(k, e, \pi_0\xi) := \gamma_1(k, e)$. Define the C^∞ function $\gamma_\rho \colon \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{(\rho-1)m} \to \mathbb{R}^m$ as follows

$$\begin{aligned}\gamma_\rho(k, e, \pi_{\rho-1}\xi) &:= \mu^{\rho-1}\gamma_{\rho-1}(k, e, \pi_{\rho-2}\xi) \\ &+ \mu^{\rho-1}\|D\gamma_{\rho-1}(k, e, \pi_{\rho-2}\xi)\|^2 \, k^4 \, (1 + \|\pi_{\rho-1}\xi\|^2)\Big(\mu^{2-\rho}\xi_{\rho-1} + \gamma_{\rho-1}(k, e, \pi_{\rho-2}\xi)\Big).\end{aligned} \tag{4.9}$$

For arbitrary $r \in \mathcal{R}$, the control strategy is given by

$$\left.\begin{aligned} u(t) &= -\gamma_\rho\big(k(t), Cx(t) - r(t), \xi(t)\big), \\ k(t) &= \frac{1}{1 - \big(\varphi(t)\|Cx(t) - r(t)\|\big)^2} \, . \end{aligned}\right\} \tag{4.10}$$

Remark 4.2.2

(i) A simple example of a function satisfying (4.5) is $\nu \colon k \mapsto k \cos k$. The rôle of the function ν is similar to the concept of a "Nussbaum" function in adaptive control. Note, however, that the requisite properties (4.5) are less restrictive than (a) the "Nussbaum property" (see Section 1.4.2) as required in [71], for example, or (b) the stronger "scaling invariant Nussbaum property", as required in [33], for example.

(ii) In the specific case of a system of relative degree $\rho = 2$, writing $e(t) = Cx(t) - r(t)$, setting $\mu = 1$ and omitting the argument t for simplicity, the control strategy takes the explicit form

$$\begin{aligned} &u = \nu(k)e - \big[(\nu'(k)\|e\|)^2 + (\nu(k))^2\big] \, k^4 \, [1 + \|\xi\|^2]\theta, \\ &k = \big[1 - \varphi^2\|e\|^2\big]^{-1}, \quad \theta = \xi - \nu(k)e, \quad \dot{\xi} = -\xi + u, \quad \xi(0) = \xi^0. \end{aligned}$$

(iii) If $CA^{\rho-1}B$ is known to be positive (respectively, negative) definite, the need for the function ν, with properties (4.5), in (4.7) is obviated and it may be replaced by $k \mapsto \nu(k) = -k$, $(k \mapsto \nu(k) = k)$, respectively.

Remark 4.2.3 Inherent conservatism in the functions γ_i for the feedback law could be improved if tighter estimates are used in the analysis; the design of k may allow for different measures of the distance to the funnel boundary. These features relate to issues of controller synthesis whilst, instead, the contribution here is considered to be analytical in nature, addressing the question of existence of controllers which guarantee performance under weak hypothesis.

4.2.3 Closed-loop system

The conjunction of (4.1), (4.4) and (4.10) defines the closed-loop initial-value problem

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + p(t,x(t)) - B\gamma_\rho(k(t), Cx(t) - r(t), \xi(t)), \quad && x(0) = x^0, \\ \dot{\xi}(t) &= F\xi(t) - G\gamma_\rho(k(t), Cx(t) - r(t), \xi(t)), && \xi(0) = \xi^0, \\ k(t) &= \frac{1}{1 - (\varphi(t)\|Cx(t) - r(t)\|)^2}\,. \end{aligned}\right\} \tag{4.11}$$

Noting the potential singularity in the function k, some care must be exercised in defining the concept of a solution of (4.11): a function $(x,\xi)\colon [0,\omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$, with $0 < \omega \leq \infty$, is deemed a *solution* of (4.11) if, and only if, it is absolutely continuous, with $(x(0),\xi(0)) = (x^0,\xi^0)$, satisfies the differential equations in (4.11) for almost all $t \in [0,\omega)$ and $\varphi(t)\,\|Cx(t) - r(t)\| < 1$ for all $t \in [0,\omega)$. A solution is *maximal* if, and only if, it has no proper right extension that is also a solution. Observe that the tracking objective is achieved if it can be shown that a solution exists and that every solution can be extended to a (maximal) solution on $\mathbb{R}_+$.

Theorem 4.2.4 *Let $(A,B,C,p) \in \mathcal{N}_\rho$ with $\rho \geq 1$ and let $\varphi \in \Phi$. For every $r \in \mathcal{R}$ and $(x^0,\xi^0) \in \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$, application of the feedback* (4.10) *in conjunction with the filter* (4.4) *to the system* (4.1) *yields the initial-value problem* (4.11) *which has a solution and every solution can be extended to a maximal solution. If a maximal solution of* (4.11) *on $[-h,\omega)$ is bounded and such that the associated gain function k is also bounded, then $\omega = \infty$.*

Proof. Define the open set

$$\mathcal{D} := \left\{(t,x,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m} \,\middle|\, \varphi(t)\|Cx - r(t)\| < 1\right\}$$

and

$$\gamma_\rho^*\colon \mathcal{D} \to \mathbb{R}^m, \quad (t,x,\xi) \mapsto \gamma_\rho\left(1/(1 - (\varphi(t)\|Cx - r(t)\|)^2),\, Cx - r(t), \xi\right),$$

the initial-value problem (4.11) may be recast on $\mathcal{D}$ as

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + p(t, x(t)) - B\gamma_\rho^*(t, x(t), \xi(t)), \\ \dot{\xi}(t) &= F\xi(t) - G\gamma_\rho^*(t, x(t), \xi(t)), \\ (0, x(0), \xi(0)) &= (0, x^0, \xi^0) \in \mathcal{D}. \end{aligned}\right\} \tag{4.12}$$

Setting $\zeta = (x, \xi)$ and defining

$$f\colon \mathcal{D} \to \mathbb{R}^{n+(\rho-1)m}$$

$$(t, \zeta) \mapsto f(t, \zeta) := \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \zeta + \begin{bmatrix} I \\ 0 \end{bmatrix} p(t, x) - \begin{bmatrix} B \\ G \end{bmatrix} \gamma_\rho^*(t, \zeta),$$

we may identify the initial-value problem (4.12) as a particular case of (3.1) as follows.

$$\dot{\zeta}(t) = f(t, \zeta(t)), \quad \zeta(0) = \zeta^0 \in \mathbb{R}^{n+(\rho-1)m}.$$

Applying Theorem 3.1.1, we conclude: (i) the existence of a solution of (4.12) and (ii) every solution can be extended to a maximal solution $(x, \xi) \in C([0, \omega), \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m})$. Furthermore, if there exists a compact set $\mathcal{K} \subset \mathcal{D}$ such that $(t, x(t), \xi(t)) \in \mathcal{K}$ for all $t \in [0, \omega)$, then $\omega = \infty$.

Clearly, a solution $(x, \xi)\colon [0, \omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ of (4.12) is also a solution of (4.11) (the converse is also true). Therefore, we may conclude that, for each $(x^0, \xi^0) \in \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$, the initial-value problem (4.11) has a solution and every solution can be maximally extended.

Let $(x, \xi)\colon [0, \omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ be a maximal solution of (4.11) and assume that (x, ξ) is bounded and that the gain function k is also bounded. Then there exist $c > 0$ and $\varepsilon > 0$ such that $\|x(t), \xi(t)\| \leq c$ and $\varphi(t)\|Cx(t) - r(t)\| \leq 1 - \varepsilon$ for all $t \in [0, \omega)$. Seeking a contradiction, suppose that $\omega < \infty$. It follows that

$$\mathcal{K} := \left\{ (t, \hat{x}, \hat{\xi}) \in \mathcal{D} \,\middle|\, \varphi(t)\|C\hat{x} - r(t)\| \leq 1 - \varepsilon,\ \|(\hat{x}, \hat{\xi})\| \leq c,\ t \in [0, \omega] \right\}$$

is a compact subset of $\mathcal{D}$ such that $(t, x(t), \xi(t))$ for all $t \in [0, \omega)$. This contradicts the final assertion of Theorem 3.1.1, and so $\omega = \infty$. □

4.3 Preliminary lemmas

We present here a series of technical lemmas that will facilitate the main result of the chapter.

Let $(A, B, C, p) \in \mathcal{N}_\rho$ with $\rho \geq 2$. Rewriting the conjunction of the system (4.1) and the filter (4.4) as

$$\left.\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} p(t, x(t)) + \begin{bmatrix} B \\ G \end{bmatrix} u(t), \\ y(t) &= [C \,\vdots\, 0] \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}, \end{aligned}\right\} \tag{4.13}$$

we prove the following technicality.

Lemma 4.3.1 *For system* (4.13)*, there exist* $K \in \mathbb{R}^{n\times(\rho-1)m}$ *and* $N \in \mathbb{R}^{(\rho-1)m\times n}$ *such that*

$$L := \begin{bmatrix} C & 0 \\ N & -NK \\ 0 & I \end{bmatrix} \in \mathbb{R}^{(n+(\rho-1)m)\times(n+(\rho-1)m)}$$

is invertible and

$$L \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} L^{-1} = \begin{bmatrix} A_1 & A_2 & \tilde{\Gamma} \\ A_3 & A_4 & 0 \\ 0 & 0 & F \end{bmatrix}, \quad L \begin{bmatrix} B \\ G \end{bmatrix} = \begin{bmatrix} 0 \\ G \end{bmatrix}, \quad [C \,\vdots\, 0]\, L^{-1} = [I \,\vdots\, 0 \,\vdots\, 0],$$

where $\tilde{\Gamma} := \left[CA^{\rho-1}B \,\vdots\, 0\right] \in \mathbb{R}^{m\times(\rho-1)m}$ *and the matrix* $A_4 \in \mathbb{R}^{(n-m)\times(n-m)}$ *is such that* $\operatorname{spec}(A_4) \subset \mathbb{C}_-$.

Proof. Define

$$K := \left[[\mu I + A]^{\rho-2}B \,\vdots\, [\mu I + A]^{\rho-3}B \,\vdots\, \cdots \,\vdots\, [\mu I + A]B \,\vdots\, B\right] \in \mathbb{R}^{n\times(\rho-1)m}$$

and note that

$$AK - KF = \left[[\mu I + A]^{\rho-1}B \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0\right], \quad KG = B \quad \text{and} \quad CK = 0.$$

Writing $\tilde{B} := (\mu I + A)^{\rho-1}B$, we have $C\tilde{B} = CA^{\rho-1}B$ and so the triple $(A, \tilde{B}, C)$ defines a linear system of relative degree one. Let $V \in \mathbb{R}^{n\times(n-m)}$ be such that $\operatorname{im} V = \ker C$.

The matrix

$$\begin{bmatrix} C \\ N \end{bmatrix}, \quad \text{with} \quad N := (V^T V)^{-1} V^T \left[I - \tilde{B}(CA^{\rho-1}B)^{-1}C\right],$$

is invertible, with inverse

$$\left[\tilde{B}(CA^{\rho-1}B)^{-1} \,\vdots\, V\right].$$

Writing

$$L = \begin{bmatrix} C & 0 \\ N & -NK \\ 0 & I \end{bmatrix} \quad \text{with} \quad L^{-1} = \begin{bmatrix} \tilde{B}(CA^{\rho-1}B)^{-1} & V & K \\ 0 & 0 & I \end{bmatrix}$$

and recalling that $KG = B$, $CB = 0$ and $CK = 0$, we have

$$L \begin{bmatrix} B \\ G \end{bmatrix} = \begin{bmatrix} 0 \\ G \end{bmatrix} \quad \text{and} \quad \left[C \,\vdots\, 0\right] L^{-1} = \left[I \,\vdots\, 0 \,\vdots\, 0\right].$$

Moreover, noting that $CAK = \left[CA^{\rho-1}B \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0\right] =: \tilde{\Gamma}$ and $N[AK - KF] = 0$, we have

$$L \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} L^{-1} = \begin{bmatrix} CA\tilde{B}(CA^{\rho-1}B)^{-1} & CAV & CAK \\ NA\tilde{B}(CA^{\rho-1}B)^{-1} & NAV & N[AK - KF] \\ 0 & 0 & F \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & \tilde{\Gamma} \\ A_3 & A_4 & 0 \\ 0 & 0 & F \end{bmatrix}.$$

It remains to show that A_4 has spectrum in open left half complex plane. Writing

$$M_1(s) = \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix} \quad \text{and} \quad M_2(s) = \begin{bmatrix} sI - A & 0 & B \\ 0 & sI - F & -G \\ C & 0 & 0 \end{bmatrix},$$

we have

$$M_3(s) := \begin{bmatrix} I & K & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} M_2(s) \begin{bmatrix} I & K & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} sI - A & AK - KF & 0 \\ 0 & sI - F & -G \\ C & 0 & 0 \end{bmatrix}.$$

In view of the particular structure of F, G and $AK - KF$, it is readily verified that

$$|\det M_3(s)| = |\det M_4(s)|, \qquad \text{where } M_4(s) = \begin{bmatrix} sI - A & [\mu I + A]^{\rho-1}B \\ C & 0 \end{bmatrix}.$$

Define

$$M_5(s) := \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} M_4(s) \begin{bmatrix} U^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} sI - A_1 & -A_2 & CA^{\rho-1}B \\ -A_3 & sI - A_4 & 0 \\ I & 0 & 0 \end{bmatrix}.$$

By the minimum-phase property of the triple (A, B, C), for all $s \in \overline{\mathbb{C}}_+$, we have $\det M_4(s) \neq 0$. We may now conclude that, for all $s \in \overline{\mathbb{C}}_+$,

$$\begin{aligned} |\det(CA^{\rho-1}B)\ \det(sI - A_4)| &= |\det M_5(s)| = |\det M_4(s)| = |\det M_3(s)| \\ &= |\det M_2(s)| = |\det(sI - F)\ \det M_1(s)| \neq 0, \end{aligned}$$

and so $\operatorname{spec}(A_4) \subset \mathbb{C}_-$. This completes the proof. □

By Lemma 4.3.1, there exist K and N such that, under the coordinate change

$$\begin{bmatrix} y(t) \\ z(t) \\ \xi(t) \end{bmatrix} = L \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}, \quad \begin{bmatrix} y^0 \\ z^0 \\ \xi^0 \end{bmatrix} = L \begin{bmatrix} x^0 \\ \xi^0 \end{bmatrix}, \quad L := \begin{bmatrix} C & 0 \\ N & -NK \\ 0 & I \end{bmatrix}, \tag{4.14}$$

the conjunction of system (4.1) and filter (4.4) is represented by

$$\left.\begin{aligned} \dot{y}(t) &= A_1 y(t) + A_2 z(t) + Cp(t, x(t)) + CA^{\rho-1}B\ \xi_1(t), \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t) + Np(t, x(t)), \\ \dot{\xi}(t) &= F\xi(t) + Gu(t), \\ (y(0), z(0), \xi(0)) &= (y^0, z^0, \xi^0) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathbb{R}^{(\rho-1)m} \end{aligned}\right\} \tag{4.15}$$

where $A_4 \in \mathbb{R}^{(n-m)\times(n-m)}$ has spectrum in open left half complex plane.

If $(x, \xi)\colon [0, \omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ is a maximal solution of the nonlinearly-perturbed closed-loop system (4.11), then, in view of (4.15) and writing

$$y(t) = Cx(t), \qquad e(t) = y(t) - r(t), \quad e^0 = y^0 - r(0), \tag{4.16}$$

we arrive at the following equivalent to (4.11),

$$\left.\begin{aligned}
\dot e(t) &= A_1 e(t) + A_2 z(t) + CA^{\rho-1}B\xi_1(t) + f_1(t), && e(0) = e^0\\
\dot z(t) &= A_3 e(t) + A_4 z(t) + f_2(t), && z(0) = z^0\\
\dot\xi(t) &= F\xi(t) - G\gamma_\rho(k(t), e(t), \xi(t)), && \xi(0) = \xi^0\\
k(t) &= 1/\big(1 - (\varphi(t)\|e(t)\|)^2\big), &&
\end{aligned}\right\} \tag{4.17}$$

where the functions f_1 and f_2 are given by

$$\left.\begin{aligned}
f_1(t) &:= A_1 r(t) + Cp(t, x(t)) - \dot r(t),\\
f_2(t) &:= A_3 r(t) + Np(t, x(t)).
\end{aligned}\right\} \tag{4.18}$$

Since $(\varphi(t)\|e(t)\|)^2 < 1$, properties of $\varphi \in \Phi$ yield boundedness of the function e which, together with boundedness of r, implies boundedness of y. By boundedness of r, essential boundedness of $\dot r$ and Assumption (A3), we may now conclude that f_1 is essentially bounded and f_2 is bounded. Now observe that, since A_4 is Hurwitz and f_2 is bounded, the second of the differential equations in (4.17) implies that z is bounded. We record these observations in the following.

Lemma 4.3.2 *Let $(A, B, C, p) \in \mathcal{N}_\rho$ with $\rho \geq 2$. Let $\mathcal{F}_\varphi$ be a performance funnel associated with $\varphi \in \Phi$. Let $r \in \mathcal{R}$ and $(x^0, \xi^0) \in \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$. If $(x, \xi)\colon [0, \omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ is a maximal solution of* (4.11), *then the functions y, z and e, given by* (4.14) *and* (4.16), *are bounded. Furthermore, the functions f_1 and f_2, given by* (4.18), *are, respectively, essentially bounded and bounded.*

The proofs of the main results (Theorems 4.4.1 and 4.4.2 below) rely crucially on a further technicality: the signals $\theta_i = \mu^{1-i}\xi_i + \gamma_i(k, e, \pi_{i-1}\xi)$, $i = 1, \ldots, \rho-1$, are bounded (and, in particular, the "mismatch" $\theta_1 = \xi_1 - \nu(k)e$ is bounded). More precisely, we have the following.

Lemma 4.3.3 *Let the hypotheses of Lemma 4.3.2 hold. If $(x, \xi)\colon [0, \omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ is a maximal solution of* (4.11), *then the signal*

$$\theta = (\theta_1, \ldots, \theta_{\rho-1})\colon [0, \omega) \to \mathbb{R}^{(\rho-1)m}$$

is bounded, where

$$\theta_i(t) = \mu^{1-i}\xi_i(t) + \gamma_i(k(t), e(t), \pi_{i-1}\xi(t)), \qquad i = 1, \ldots, \rho-1, \tag{4.19}$$

with the notational convention $\gamma_1(k, e, \pi_0\xi) := \gamma_1(k, e)$.

Proof. Assume that $(x,\xi)\colon [0,\omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ is a maximal solution of (4.11). Write $y(t) = Cx(t)$ and $e(t) = y(t) - r(t)$ for all $t \in [0,\omega)$. By Lemma 4.3.1, there exists an invertible linear transformation L under which the closed-loop system (4.11) may be expressed in the form (4.17), wherein, by Lemma 4.3.2, e and z are bounded and the functions f_1 and f_2, given by (4.18), are, respectively, essentially bounded and bounded. By the first of equations (4.17), we may infer the existence of $c_1 > 0$ such that

$$\|\dot e(t)\| \le c_1\big(1 + \|\xi_1(t)\|\big) \quad \text{for a.a. } t \in [0,\omega).$$

By boundedness of φ, e and essential boundedness of $\dot\varphi$, there exists $c_2 > 0$ such that

$$\left.\begin{aligned} |\dot k(t)| &= 2k^2(t)\big|\varphi^2(t)\langle e(t), \dot e(t)\rangle + \varphi(t)\dot\varphi(t)\|e(t)\|^2\big| \\ &\le c_2\, k^2(t)\,(1 + \|\xi_1(t)\|) \quad \text{for a.a. } t \in [0,\omega). \end{aligned}\right\} \tag{4.20}$$

Since $k(t) \ge 1$ for all $t \in [0,\omega)$, we may now conclude the existence of $c_3 > 0$ such that

$$\|(\dot k(t), \dot e(t))\|^2 \le c_3 \Delta(t) \quad \text{where } \Delta(t) := k^4(t)\big(1 + \|\xi_1(t)\|^2\big).$$

Then, writing $c_{4,1} = c_3/\mu > 0$ and invoking (4.8), we have

$$\begin{aligned} \langle \theta_1(t), \dot\theta_1(t)\rangle &\le \langle \theta_1(t)\,,\, -\mu\xi_1(t) + \xi_2(t)\rangle + \|\theta_1(t)\|\|D\gamma_1(k(t),e(t))\|\|(\dot k(t), \dot e(t))\| \\ &\le \langle \theta_1(t)\,,\, -\mu\theta_1(t) + \mu\gamma_1(k(t),e(t))\rangle + \langle \theta_1(t), \xi_2(t)\rangle \\ &\qquad + \sqrt{\mu}\,\|\theta_1(t)\|\,\|D\gamma_1(k(t),e(t))\|\,\sqrt{(c_3/\mu)\,\Delta(t)} \\ &\le c_{4,1} - \mu\|\theta_1(t)\|^2 + \langle \theta_1(t), \xi_2(t)\rangle + \mu\langle \theta_1(t), \gamma_1(k(t),e(t))\rangle \\ &\qquad + \mu\|\theta_1(t)\|^2\,\|D\gamma_1(k(t),e(t))\|^2\,\Delta(t) \\ &= c_{4,1} - \mu\|\theta_1(t)\|^2 + \big\langle \theta_1(t), \xi_2(t) + \mu\gamma_2(k(t),e(t),\xi_1(t))\big\rangle \\ &= c_{4,1} - \mu\|\theta_1(t)\|^2 + \mu\langle \theta_1(t), \theta_2(t)\rangle \qquad \text{for a.a. } t \in [0,\omega). \end{aligned}$$

Analogous calculations yield the existence of constants $c_{4,2}, \ldots, c_{4,\rho-1} > 0$, such that

$$\langle \theta_i(t), \dot\theta_i(t)\rangle \le c_{4,i} - \mu\|\theta_i(t)\|^2 + \mu\langle \theta_i(t), \theta_{i+1}(t)\rangle \quad \text{a.a. } t \in [0,\omega),\ \ i = 2, \ldots, \rho-2$$

and

$$\langle \theta_{\rho-1}(t), \dot\theta_{\rho-1}(t)\rangle \le c_{4,\rho-1} - \mu\|\theta_{\rho-1}(t)\|^2 \quad \text{for a.a. } t \in [0,\omega).$$

Writing $c_4 = c_{4,1} + \ldots + c_{4,\rho-1}$, we have

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}\|\theta(t)\|^2 &\leq c_4 - \mu\|\theta(t)\|^2 + \mu\langle\theta_1(t), \theta_2(t)\rangle + \cdots + \mu\langle\theta_{\rho-2}(t), \theta_{\rho-1}(t)\rangle \\ &= c_4 - \mu\langle\theta(t), P\theta(t)\rangle \qquad \text{for a.a. } t \in [0,\omega),\end{aligned}$$

where P is a positive-definite, symmetric, tridiagonal matrix with all diagonal entries equal to 1 and all sub- and superdiagonal entries equal to -1/2 (in fact, P is the symmetric part of F). By positivity of P, it follows that θ is bounded. This completes the proof of the lemma. □

4.4 Main results

4.4.1 Relative degree $\rho \geq 2$ case

Firstly, we consider systems of relative degree $\rho \geq 2$.

Theorem 4.4.1 *Let $(A, B, C, p) \in \mathcal{N}_\rho$ with $\rho \geq 2$ and let $\mathcal{F}_\varphi$ be a performance funnel associated with $\varphi \in \Phi$. For every $r \in \mathcal{R}$ and $(x^0, \xi^0) \in \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$, application of the feedback* (4.10) *in conjunction with the filter* (4.4) *to system* (4.1) *yields the initial-value problem* (4.11) *which has a solution and every solution can be extended to a maximal solution. Every maximal solution $(x, \xi)\colon [0,\omega) \to \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ has the properties:*

(i) $\omega = \infty$;

(ii) *all variables (x, ξ), k and u are bounded;*

(iii) *the tracking error evolves within the funnel $\mathcal{F}_\varphi$ and is bounded away from the funnel boundary, i.e. there exists $\varepsilon > 0$ such that, for all $t \geq 0$, $\varphi(t)\,\|Cx(t) - r(t)\| \leq 1 - \varepsilon$.*

Proof. Let $(x^0, \xi^0) \in \mathbb{R}^n \times \mathbb{R}^{(\rho-1)m}$ be arbitrary. By Theorem 4.2.4, (4.11) has a solution and every solution can be maximally extended. Let (x, ξ) be a maximal solution of (4.11) with interval of existence $[0, \omega)$. Writing $y(t) = Cx(t)$, $e(t) = y(t) - r(t)$ for all $t \in [0,\omega)$ and invoking Lemma 4.3.1, there exists an invertible linear transformation L which takes (4.11) into the equivalent form (4.17)–(4.18). Introducing $\theta_1\colon [0,\omega) \to \mathbb{R}^m$ given by (4.19), viz.

$$\theta_1(t) = \xi_1(t) - \nu(k(t))e(t),$$

then, by the first of equations (4.17), we have

$$\dot{e}(t) = f_3(t) + \nu(k(t))\, CA^{\rho-1}B\, e(t) \quad \text{for a.a. } t \in [0,\omega), \tag{4.21}$$

with

$$f_3(t) := A_1 e(t) + A_2 z(t) + CA^{\rho-1} B\theta_1(t) + f_1(t).$$

By Lemmas 4.3.2 and 4.3.3, the functions y, z, e, and $\theta = (\theta_1, ..., \theta_{\rho-1})$, given by (4.19), are bounded which, together with essential boundedness of f_1, implies essential boundedness of f_3. Therefore, there exists $c_5 > 0$ such that

$$\langle e(t), \dot{e}(t)\rangle \leq c_5 + \nu(k(t)) \langle e(t), CA^{\rho-1}Be(t)\rangle \quad \text{for a.a. } t \in [0, \omega). \tag{4.22}$$

We are now in a position to prove boundedness of k. Writing

$$\beta_0 := \frac{1}{2} \left\| \left((CA^{\rho-1}B)^T + CA^{\rho-1}B\right)^{-1} \right\|^{-1}, \qquad \beta_1 := \|CA^{\rho-1}B\|$$

and recalling that $CA^{\rho-1}B$ is either positive definite or negative definite, we have

$$\beta_0 \|e\|^2 \leq |\langle e, CA^{\rho-1}Be\rangle| \leq \beta_1 \|e\|^2 \quad \forall\, e \in \mathbb{R}^m.$$

Define the continuous function $\tilde{\nu}\colon \mathbb{R} \to \mathbb{R}$ as follows.

Case (a): If $CA^{\rho-1}B$ is positive definite, then set

$$\tilde{\nu}(k) := \begin{cases} -\beta_1 \nu(k), & \nu(k) \geq 0, \\ -\beta_0 \nu(k), & \nu(k) < 0. \end{cases}$$

Case (b): If $CA^{\rho-1}B$ is negative definite, then set

$$\tilde{\nu}(k) := \begin{cases} \beta_0 \nu(k), & \nu(k) \geq 0, \\ \beta_1 \nu(k), & \nu(k) < 0. \end{cases}$$

Therefore,

$$\nu(k)\langle e, CA^{\rho-1}Be\rangle \leq -\tilde{\nu}(k)\|e\|^2 \quad \forall\, e \in \mathbb{R}^m \quad \forall\, k \geq 0,$$

which, together with boundedness of e, φ, essential boundedness of $\dot{\varphi}$ and (4.22), implies the existence of $c_6 > 0$ such that

$$\begin{aligned} \frac{d}{dt}(\varphi(t)\|e(t)\|)^2 &= 2\varphi(t)\dot{\varphi}(t)\|e(t)\|^2 + 2\varphi^2(t)\langle e(t), \dot{e}(t)\rangle \\ &\leq c_6 - 2\,\varphi^2(t)\,\tilde{\nu}(k(t))\,\|e(t)\|^2 \quad \text{for a.a. } t \in [0, \omega). \end{aligned}$$

By properties (4.5) of ν, there exists a strictly increasing unbounded sequence (k_j) in $(1, \infty)$ such that $(\tilde{\nu}(k_j))$ is also unbounded and strictly increasing. Seeking a contradiction, suppose that k is unbounded. For each $j \in \mathbb{N}$, define

$$\begin{aligned}
\tau_j &:= \inf\{t \in [0,\omega) |\ k(t) = k_{j+1}\}, \\
\sigma_j &:= \sup\{t \in [0,\tau_j] |\ \tilde{\nu}(k(t)) = \tilde{\nu}(k_j)\}, \\
\tilde{\sigma}_j &:= \sup\{t \in [0,\tau_j] |\ k(t) = k_j\} \le \sigma_j .
\end{aligned}$$

It is readily verified that $\sigma_j < \tau_j$ and $k(\sigma_j) < k(\tau_j)$. Then, for all $j \in \mathbb{N}$ and all $t \in [\sigma_j, \tau_j]$, we have $k(t) \ge k_j$ and $\tilde{\nu}(k(t)) \ge \tilde{\nu}(k_j)$. Therefore,

$$(\varphi(t)\|e(t)\|)^2 \ge 1 - \frac{1}{k_j} \ge 1 - \frac{1}{k_1} =: c_7 > 0 \quad \forall\, t \in [\sigma_j, \tau_j] \quad \forall\, j \in \mathbb{N}$$

and so

$$\frac{d}{dt}(\varphi(t)\|e(t)\|)^2 \le c_6 - 2c_7\, \tilde{\nu}(k(t)) \quad \forall\, t \in [\sigma_j, \tau_j] \quad \forall\, j \in \mathbb{N}.$$

Let $j^* \in \mathbb{N}$ be sufficiently large so that $c_6 - 2c_7\tilde{\nu}(k_{j^*}) < 0$. Then,

$$\big(\varphi(\tau_{j^*})\|e(\tau_{j^*})\|\big)^2 - \big(\varphi(\sigma_{j^*})\|e(\sigma_{j^*})\|\big)^2 < 0,$$

whence the contradiction

$$0 > \frac{1}{1 - \big(\varphi(\tau_{j^*})\|e(\tau_{j^*})\|\big)^2} - \frac{1}{1 - \big(\varphi(\sigma_{j^*})\|e(\sigma_{j^*})\|\big)^2} = k(\tau_{j^*}) - k(\sigma_{j^*}) \ge 0.$$

This proves boundedness of k.

Next we show boundedness of ξ, x and u. Since k is bounded, there exists $\varepsilon > 0$ such that $\varphi(t)\|e(t)\| \le 1 - \varepsilon$ for all $t \in [0,\omega)$. By boundedness of y, z, θ and k, it follows from the recursive construction in (4.19) that, for $i = 1, ..., \rho-1$, γ_i and ξ_i are bounded. Consequently x is bounded and, by (4.7) and (4.8), boundedness of γ_ρ (and hence of u) follows. Finally, by boundedness of x, ξ and k, together with Theorem 4.2.4, we may conclude that $\omega = \infty$. This completes the proof of the theorem. □

4.4.2 Relative degree $\rho = 1$ case

Secondly, we consider the case wherein the triple (A, B, C) defines a minimum-phase system of relative degree $\rho = 1$. In this case, a filter is not necessary and the controller simplifies to

$$u(t) = \nu(k(t))(Cx(t) - r(t)), \quad k(t) = \frac{1}{1 - \big(\varphi(t)\|Cx(t) - r(t)\|\big)^2}. \tag{4.23}$$

The closed-loop initial-value problem then becomes

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + p(t, x(t)) + B\nu(k(t))(Cx(t) - r(t)), \quad x(0) = x^0, \\ k(t) &= \frac{1}{1 - (\varphi(t)\|Cx(t) - r(t)\|)^2}. \end{aligned}\right\} \tag{4.24}$$

In the case of sign-definite CB of known sign, the result of Theorem 4.4.2 is proved in [26]; the general case of Theorem 4.4.2, wherein CB is of unknown sign, is new.

Theorem 4.4.2 *Let $(A, B, C, p) \in \mathcal{N}_1$ and let $\mathcal{F}_\varphi$ be a performance funnel associated with $\varphi \in \Phi$. For every $r \in \mathcal{R}$ and $x^0 \in \mathbb{R}^n$, the initial-value problem (4.24) has a solution and every solution can be extended to a maximal solution. Every maximal solution $x\colon [0, \omega) \to \mathbb{R}^n$ has the properties:*

(i) $\omega = \infty$;

(ii) *x, k and u are bounded;*

(iii) *there exists $\varepsilon > 0$ such that, for all $t \geq 0$, $\varphi(t)\, \|Cx(t) - r(t)\| \leq 1 - \varepsilon$.*

Proof. This is a straightforward modification of the proof of Theorem 4.4.1, essentially excising all vestiges of the filter equations. □

4.5 Example

We illustrate the controller strategy (4.10) for the single-input, single-output, relative degree two system with nonlinear perturbations, introduced in Section 1.1.3, modelling a pendulum (with input force u):

$$\ddot{y}(t) + a \sin y(t) = b\, u(t)\,, \tag{4.25}$$

with unknown real parameters a and $b \neq 0$. Equation (4.25) is equivalent to (4.1) with $x(t) = (y(t), \dot{y}(t))^T$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ b \end{bmatrix}, \; C = [1 \,\vdots\, 0], \quad p(t, x(t)) = a \sin y(t), \; t \geq 0.$$

The funnel is specified by the smooth function

$$t \mapsto \varphi(t) = \begin{cases} 10(1 - (0.1\,t - 1)^2), & 0 \leq t < 10, \\ 10, & t \geq 10, \end{cases} \tag{4.26}$$

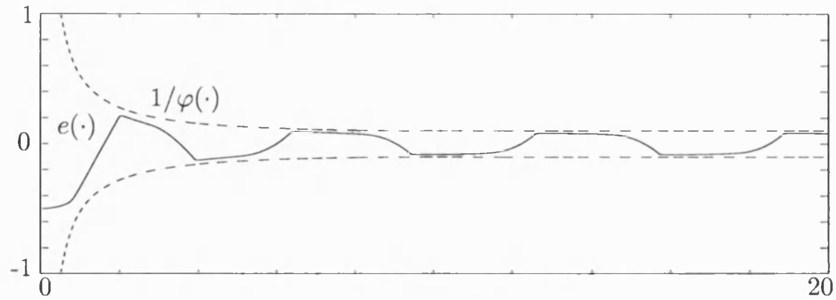
which assures a tracking accuracy $|e(t)| < 0.1$ for all $t \geq 10$. If non-zero b is of unknown sign, then, choosing $\nu\colon k \mapsto k\cos k$, writing $e(t) = y(t) - r(t)$ and suppressing the argument t for simplicity, the control strategy is

$$\left.\begin{aligned} u &= \mu(k\cos k)e \\ &\quad -\mu\big[\xi - (k\cos k)e\big]\big[(\cos k - k\sin k)^2 e^2 + k^2\cos^2 k\big]\, k^4\,[1+\xi^2], \\ k &= \big[1-\varphi^2 e^2\big]^{-1}, \\ \dot{\xi} &= -\mu\xi + u, \qquad\qquad \xi(0) = 0. \end{aligned}\right\} \tag{4.27}$$

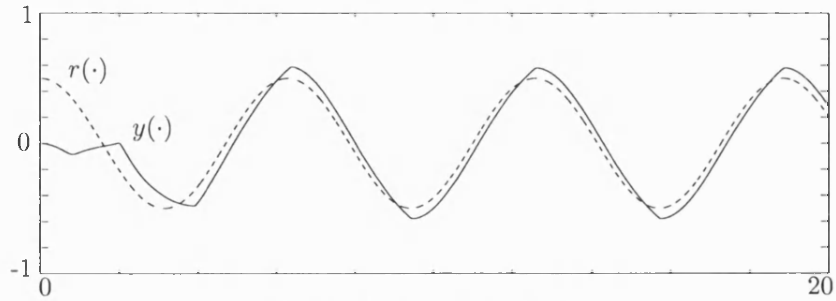
Adopting the values $a = \frac{1}{2}$, $b = 1$, $\mu = 1$, initial data $(y(0), \dot{y}(0)) = (0, 0)$ and reference signal $t \mapsto r(t) = \frac{1}{2}\cos t$, the behaviour of the closed-loop system (4.25)-(4.27) over the time interval $[0, 20]$ is depicted in Figure 4-3. The "peaks" in the control action occur whenever the tracking error is close to the boundary of the funnel. However, if $b \neq 0$ is known *a priori* to be positive, then the peaking behaviour is considerably mollified by choosing the function $\nu\colon k \mapsto -k$ in place of $k \mapsto k\cos k$, in which case the strategy is

$$\left.\begin{aligned} u &= -ke - [\xi + ke]\big[e^2 + k^2\big]\, k^4\,[1+\xi^2] \\ k &= \big[1-\varphi^2 e^2\big]^{-1} \\ \dot{\xi} &= -\xi + u, \qquad\qquad \xi(0) = 0. \end{aligned}\right\} \tag{4.28}$$

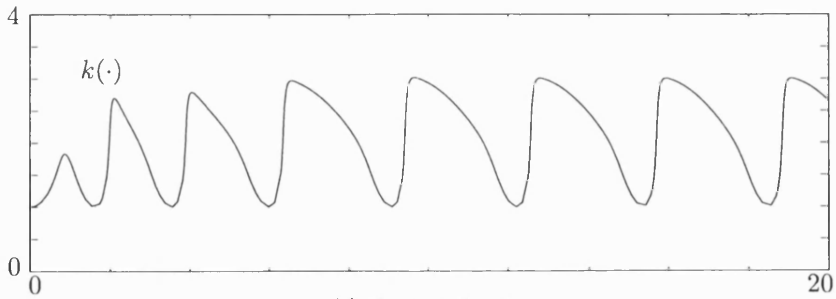
For the same parameter values and initial data as above, the behaviour (4.25), under control (4.28), is shown in Figure 4-4.



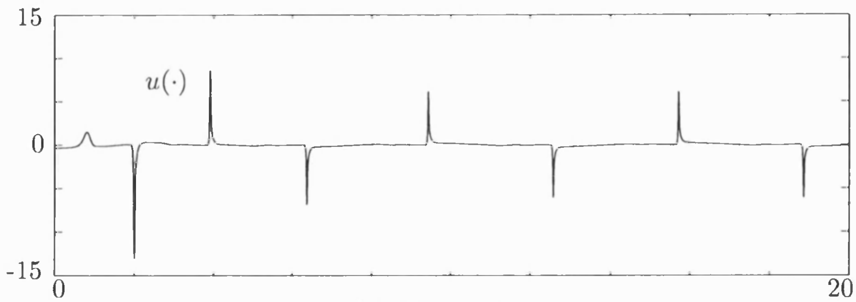
(a) The funnel and tracking error e



(b) The reference r and output y

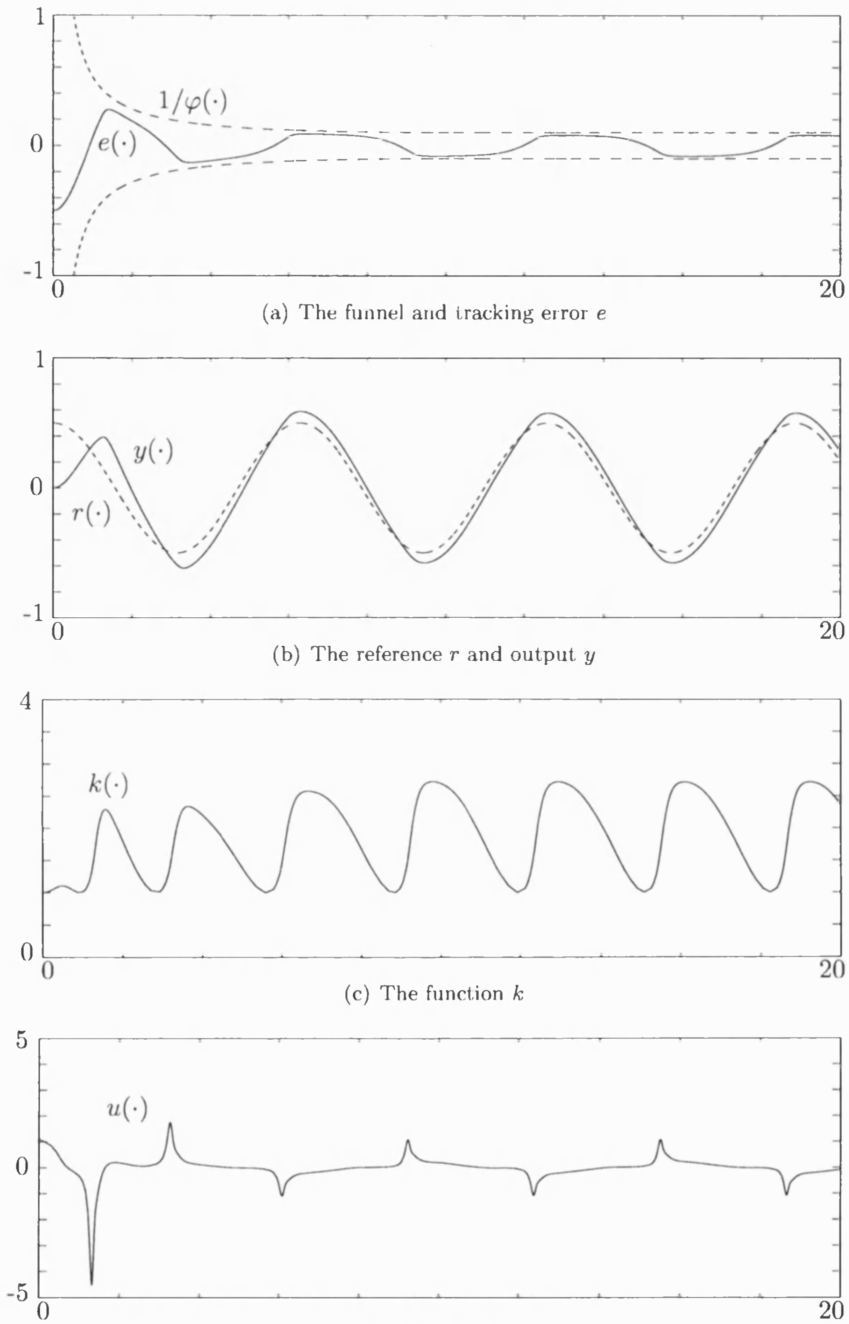


(c) The function k



(d) The control u

Figure 4-3: Unknown sign $b \neq 0$: control (4.27) applied to the nonlinear pendulum (4.25).



(a) The funnel and tracking error e

(b) The reference r and output y

(c) The function k

(d) The control u

Figure 4-4: Known sign $b > 0$: control (4.28) applied to the nonlinear pendulum (4.25).

Chapter 5

Approximate tracking for nonlinear systems with known relative degree

5.1 Introduction

In Chapter 4, an approximate tracking objective, with prescribed transient behaviour, was achieved for a class of nonlinearly-perturbed systems of the form

$$
\begin{aligned}
\dot{x}(t) &= A\,x(t) + B\,u(t) + p(t, x(t)), \qquad x(0) = x^0 \in \mathbb{R}^n, \\
y(t) &= C\,x(t) \in \mathbb{R}^m,
\end{aligned}
$$

under the assumptions (A1)–(A3) listed in Section 4.1.1. The main results of Chapter 4 can be extended to a larger class of nonlinear systems that invoke the class of operators introduced in Chapter 2. In particular, we consider approximate tracking of a reference signal in the context of a class of multi-input, multi-output dynamical systems, modelled by functional differential equations. The system class encompasses a wide variety of nonlinear effects, including hysteresis phenomena and delays.

This work is also a natural extension to that of [26], in which a class of infinite-dimensional, m-input ($u(t) \in \mathbb{R}^m$), m-output ($y(t) \in \mathbb{R}^m$), nonlinear systems (with finite memory) given by a controlled functional differential equation of the form

$$
\dot{y}(t) = g(p(t), (Ty)(t), u(t)), \quad y|_{[-h,0]} = y^0 \in C([-h, 0], \mathbb{R}^m),
$$

is considered, where g is a continuous function, p represents a bounded perturbation and T is a causal operator of class $\mathcal{T}_h^m$. An output feedback control structure is developed which ensures approximate asymptotic tracking, with prescribed transient behaviour, of any reference signal of class $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$. Here, we extend these investigations to incorporate higher-order systems, affine in the control, of the form

$$y^{(\rho)}(t) = R_1\, y(t) + R_2\, y^{(1)}(t) + \cdots + R_\rho\, y^{(\rho-1)}(t) + g(p(t), (Ty)(t)) + \Gamma u(t) \qquad (5.1)$$

where $\rho \in \mathbb{N}$ is known, $y^{(i)}$ denotes the ith derivative of y and the matrix Γ is assumed to be sign definite (equivalently, $\langle v, \Gamma v\rangle = 0 \;\Leftrightarrow\; v = 0$). The structure for the system implementing an error feedback strategy, where the error is the difference between the output y and a reference signal r of class $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$, is illustrated in Figure 5-1.

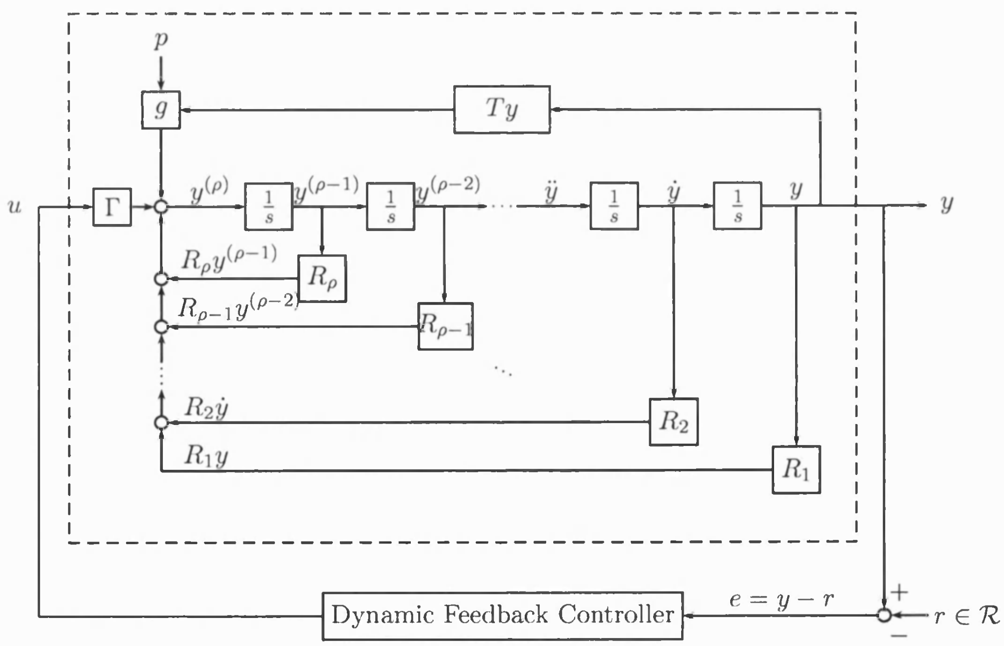


Figure 5-1: Structure of system (5.1).

This chapter is based on joint work with A. Ilchmann and E. P. Ryan in [28] and is organized as follows. Sections 5.1.1 and 5.2 introduce the control objectives and the system class: Section 5.3 highlights several particular subclasses. In Section 5.4, the control and feedback laws are constructed. The existence theory developed in Chapter 3, see Theorem 3.1.1, will then be applied to the resulting closed-loop system in Section 5.4.3. The main results on transient and asymptotic behaviour of the closed-loop are given in Section 5.6 and illustrated in an example in Section 5.7.

5.1.1 Control objectives and the performance funnel

The two control objectives considered coincide with those of Chapter 4:

(i) approximate tracking, by the output y, of reference signals r of class $\mathcal{R} := W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$;

(ii) prescribed transient behaviour of the tracking error signal.

As before, both objectives are captured by a performance funnel, see Definition 1.3.1,

$$\mathcal{F}_\varphi = \left\{(t,e) \in \mathbb{R}_+ \times \mathbb{R}^m \middle| \, \varphi(t)\, \|e\| < 1\right\}$$

associated with a function φ belonging to

$$\Phi = \left\{\varphi \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \;\middle|\; \varphi(0) = 0, \quad \varphi(s) > 0 \text{ for all } s > 0 \text{ and } \liminf_{s \to \infty} \varphi(s) > 0\right\}.$$

The aim is an output feedback strategy ensuring that, for every reference signal $r \in \mathcal{R}$, the tracking error $e = y - r$ evolves within the funnel $\mathcal{F}_\varphi$. As in Chapter 4, comparisons can be made with the early contribution by Miller and Davison [48], in which the attainment of prescribed transient behaviour for a class of single-input, single-output, linear, minimum-phase systems with known high-frequency gain is considered. The control strategy in [48] is adaptive with non-decreasing gain k and is less flexible in its scope for shaping the transient behaviour.

The feedback structure implemented in this chapter mirrors that of the controller in Chapter 4; it essentially exploits an intrinsic high-gain property of the system/filter interconnection by ensuring that, if $(t, e(t))$ approaches the funnel boundary, then the gain attains values sufficiently large to preclude boundary contact. Boundedness of φ means that an exact asymptotic tracking objective cannot be enforced. However, in the specific case of relative degree 1 systems, an exact asymptotic tracking objective will be considered in Chapters 6 and 7.

5.2 Class of systems

We subsume (5.1) in the following

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + f(p(t), (Ty)(t), x(t)) + Bu(t), \\ y(t) &= Cx(t), \\ x|_{[-h,0]} &= x^0 \in C([-h,0], \mathbb{R}^{\rho m}), \end{aligned}\right\} \tag{5.2}$$

with

$$A = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho \end{bmatrix} \in \mathbb{R}^{\rho m \times \rho m}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma \end{bmatrix} \in \mathbb{R}^{\rho m \times m}, \tag{5.3}$$

$$C = \quad [I \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0 \,\vdots\, 0] \in \mathbb{R}^{m \times \rho m}, \quad f \colon \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^{\rho m} \to \mathbb{R}^{\rho m} \text{ continuous.} \tag{5.4}$$

Observe that $\Gamma = CA^{\rho-1}B$. In the special case wherein f is given by

$$f(p, w, x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g(p, w) \end{bmatrix}, \tag{5.5}$$

it is clear that (5.1) and (5.2) are equivalent.

Recalling Definition 2.1.1, in which the class of operators $\mathcal{T}_h^m$ was introduced, we now formally define the class of systems considered in this chapter.

Definition 5.2.1 (System class Σ_ρ)
For $\rho \in \mathbb{N}$, Σ_ρ is the class of m-input, m-output systems (A, B, C, f, p, T, h) of the form (5.2), where $h \geq 0$ quantifies the memory of the system, A, B and C are structured as in (5.3)-(5.4) and satisfy

(B1) *sign-definite high-frequency gain: $\Gamma = CA^{\rho-1}B$ is either positive definite or negative definite (equivalently, $\langle v, \Gamma v\rangle = 0 \Leftrightarrow v = 0$).*

The functions f, p and operator T are such that

(B2) $p \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$, (B3) *for some* $q \in \mathbb{N}$, $T\colon C([-h,\infty),\mathbb{R}^m) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+,\mathbb{R}^q)$ *is of class* $\mathcal{T}_h^m$,

(B4) $f\colon \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^{\rho m} \to \mathbb{R}^{\rho m}$ *is continuous and, for all non-empty compact sets* $P \subset \mathbb{R}^m$, $W \subset \mathbb{R}^q$ *and* $Y \subset \mathbb{R}^m$, *there exists a constant* $c_0 = c_0(P,W,Y) > 0$ *such that*

$$\|f(p,w,x)\| \le c_0 \ \textit{for all}\ (p,w,x) \in P \times W \times \{v \in \mathbb{R}^{\rho m} \mid Cv \in Y\}.$$

Remark 5.2.2

(i) Observe that

$$CA^iB = 0 \quad \text{for } i = 1,...,\rho-2 \text{ and } \Gamma = CA^{\rho-1}B \ \text{ is invertible.} \tag{5.6}$$

Therefore, recalling the definition of relative degree from Section 1.4.6, it follows from (5.6) that, in the case $f \equiv 0$, the linear system (A,B,C) is said to have relative degree ρ. Note that Assumption (B1) requires the strengthened assumption that $CA^{\rho-1}B$ is either positive or negative definite. In the multi-input, multi-output case, (B1) is rather restrictive, though we emphasize that symmetry of $CA^{\rho-1}B$ is not required. By contrast, in the single-input, single-output case, the assumption of sign definiteness is redundant and (B1) is simply equivalent to positing that the relative degree of the linear triple (A,B,C) is known.

(ii) Recalling the definition of a minimum-phase system in Section 1.4.4, observe that, due to the structure of the matrices A, B and C in (5.3)–(5.4) and Assumption (B1), the linear system (A,B,C) is minimum phase.

(iii) Assumption (B4) constrains the nature of the dependence of f on its third argument: in particular, for compact sets P, W and Y, it posits boundedness of f on $P \times W \times C^{-1}(Y)$. For example, (B4) holds if there exists a continuous function $\pi\colon \mathbb{R}^m \times \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}_+$ such that $\|f(p,w,x)\| \le \pi(p,w,Cx)$ for all (p,w,x). Assumption (B4) plays a crucial role in the later analysis: in its absence (i.e. if f is merely assumed to be continuous), the performance objectives cannot be achieved (indeed, finite escape times can occur). For example, consider the single-input, single-output, relative-degree-two system of the form (5.2), with $R_1, R_2 = 0$, $p \equiv 0$, $T \equiv 0$, $\Gamma = 1$ and f given by $f(x_1,x_2) = [x_1^2 - x_2 \quad 0]^T$, as follows.

$$\dot{x}_1(t) = x_1^2(t), \qquad \dot{x}_2(t) = u.$$

Assumptions (B1)–(B3) are trivially satisfied, but with Assumption (B4) relaxed, in this example, the output x_1 cannot be influenced by the controller and will exit the performance funnel in finite time.

(iv) The paper [6], discussed in Section 1.6, considers stabilization and tracking for a class of relative-degree-one nonlinear systems and it is stated that the main results can be achieved for higher relative degree systems by means of a semiglobal backstepping lemma. The multi-layered nature of the assumptions determining the system class considered in [6] makes it difficult to assess the overlap with the class of systems Σ_ρ.

(v) With reference to Figure 5-2, the system (5.2) can be thought of as the interconnection of two blocks. The dynamical system represented by block Λ_1, which can be influenced directly by the system control u, is also driven by the output w from the dynamic block Λ_2, as shown in Figure 5-2. The block Λ_2 can be considered as a causal operator mapping the system output y to w (an internal quantity, unavailable for feedback purposes); it allows for infinite-dimensional (e.g. delays, diffusions) and hysteresis (e.g. backlash) effects, as discussed in Chapter 2.

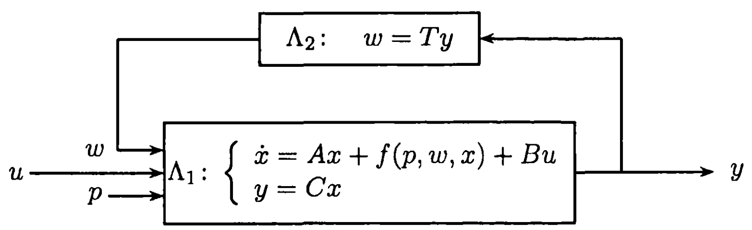


Figure 5-2: System of class Σ_ρ.

5.3 Subclasses of Σ_ρ

5.3.1 Finite-dimensional linear prototype

For motivational purposes and comparison with the system class $\mathcal{N}_\rho$, considered in Chapter 4, we examine a prototype linear system and show that all finite-dimensional linear systems of this form are incorporated in the class Σ_ρ. Consider an m-input, m-output linear system of the form

$$\dot{w}(t) = \tilde{A}\, w(t) + \tilde{B}\, u(t), \quad w(0) = w^0 \in \mathbb{R}^n, \quad y(t) = \tilde{C}\, w(t), \tag{5.7}$$

with relative degree $\rho \geq 1$, $\tilde{A} \in \mathbb{R}^{n\times n}$, $\tilde{B} \in \mathbb{R}^{n\times m}$, $\tilde{C} \in \mathbb{R}^{m\times n}$, $n \geq \rho m$ and positive-definite or negative-definite high-frequency gain $\tilde{C}\tilde{A}^{\rho-1}\tilde{B}$. Note that this prototype class of systems is equivalent to $\mathcal{N}_\rho$ in the case when the nonlinear perturbation in Assumption (A3) of $\mathcal{N}_\rho$ is identically zero. To show that the system (5.7) belongs to the class Σ_ρ, we present the following lemma.

Lemma 5.3.1 *Consider a linear system of the form* (5.7) *with relative degree* $\rho \in \mathbb{N}$. *Define*

$$\mathcal{C} := \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{\rho-1} \end{bmatrix} \in \mathbb{R}^{\rho m\times n}, \qquad \mathcal{B} := [\tilde{B} \,\vdots\, \tilde{A}\tilde{B} \,\vdots\, \cdots \,\vdots\, \tilde{A}^{\rho-1}\tilde{B}] \in \mathbb{R}^{n\times \rho m}$$

and let $\mathcal{V} \in \mathbb{R}^{n\times(n-\rho m)}$ *be such that* $\operatorname{im} \mathcal{V} = \ker \mathcal{C}$. *Then*

(i) $\mathbb{R}^n = \ker \mathcal{C} \oplus \operatorname{im} \mathcal{B}$;

(ii) *the matrix*

$$\mathcal{U} = \begin{bmatrix} \mathcal{C} \\ \mathcal{N} \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad \text{where} \quad \mathcal{N} = (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T[I - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}] \in \mathbb{R}^{(n-\rho m)\times n},$$

is invertible, with inverse $\mathcal{U}^{-1} = [\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \,\vdots\, \mathcal{V}]$, *and the triple*

$$(\hat{A}\,,\,\hat{B}\,,\,\hat{C}) := (U\tilde{A}U^{-1},\, U\tilde{B}\,,\, \tilde{C}U^{-1}) \tag{5.8}$$

has the following structure (wherein I *and* 0 *denote the* $m \times m$ *identity matrix and zero matrix, respectively)*

$$\hat{A} = \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & I & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S \\ P & 0 & \cdots & 0 & 0 & Q \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma \\ 0 \end{bmatrix}, \quad \hat{C} = [I \,\vdots\, 0 \,\vdots\, \cdots \,\vdots\, 0 \,\vdots\, 0 \,\vdots\, 0], \tag{5.9}$$

with $[R_1 \,\vdots\, \cdots \,\vdots\, R_\rho \,\vdots\, S] = \tilde{C}\tilde{A}^\rho\mathcal{U}^{-1}$, $\Gamma = \tilde{C}\tilde{A}^{\rho-1}\tilde{B}$, $P = \mathcal{N}\tilde{A}^\rho\tilde{B}\Gamma^{-1}$, *and* $Q = \mathcal{N}\tilde{A}\mathcal{V}$;

(iii) *if the system* (5.7) *is minimum phase, then* $\operatorname{spec}(Q) \subset \mathbb{C}_-$.

We remark that, in the case $\rho = 1$, (5.9) is to be interpreted as

$$\hat{A} = \begin{bmatrix} R_1 & S \\ P & Q \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \Gamma \\ 0 \end{bmatrix}, \quad \hat{C} = [I \,\vdots\, 0]. \tag{5.10}$$

Parts of the following proof are implicit in the proofs of [31, Lemma 4.1.1] and [32, Propositions 11.5.1 and 11.5.2] (in a general context of nonlinear systems); here, we provide a simple, self-contained proof in the restricted context of linear systems.

Proof. STEP (i): First note that

$$\mathcal{C}\mathcal{B} = \begin{bmatrix} 0 & & \Gamma \\ & \cdot^{\cdot^{\cdot}} & \\ \Gamma & & * \end{bmatrix}$$

and, since Γ is invertible, we see that $\mathcal{C}\mathcal{B} \in GL_{\rho m}(\mathbb{R})$. Furthermore, $\mathcal{N}\mathcal{B} = 0$. Assertion (i) then follows from the observation that, for any $x \in \mathbb{R}^n$, we have

$$v := (I - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C})x \in \ker \mathcal{C} \quad \text{and} \quad w := \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}x \in \operatorname{im} \mathcal{B},$$

and so $x = v + w$.

STEP (ii): We now prove Assertion (ii). It is clear that $\mathcal{U}^{-1} = [\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \,\vdots\, \mathcal{V}]$. It is also immediate that $\hat{B} := \mathcal{U}\tilde{B}$ and $\hat{C} := \tilde{C}\mathcal{U}^{-1}$ have the structure given in (5.9). Furthermore ,

$$\mathcal{U}\tilde{A} = \hat{A}\mathcal{U} \tag{5.11}$$

for some $\hat{A}$ of the form:

$$\hat{A} = \begin{bmatrix} 0 & I & 0 & \dots & 0 & 0 \\ 0 & 0 & I & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & I & 0 \\ R_1 & R_2 & \dots & R_{\rho-1} & R_\rho & S \\ P_1 & P_2 & \dots & P_{\rho-1} & P_\rho & Q \end{bmatrix},$$

with $R_i \in \mathbb{R}^{m \times m}$, $P_i \in \mathbb{R}^{(n-\rho m)\times m}$, $i = 1, \dots, \rho$, $S \in \mathbb{R}^{m \times (n-\rho m)}$, $Q = \mathcal{N}\tilde{A}\mathcal{V} \in \mathbb{R}^{(n-\rho m)\times(n-\rho m)}$ and $[R_1 \,\vdots\, \cdots \,\vdots\, R_\rho \,\vdots\, S] = \tilde{C}\tilde{A}^\rho\mathcal{U}^{-1}$. If $\rho = 1$, then $\hat{A}$ takes the form shown in (5.10).

Recalling that $\mathcal{N}\mathcal{B} = 0$, we see that

$$[P_1 \vdots \cdots \vdots P_\rho] = \mathcal{N}\tilde{A}\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} = [0 \vdots \cdots \vdots 0 \vdots \mathcal{N}\tilde{A}^\rho \tilde{B}] \begin{bmatrix} * & & \Gamma^{-1} \\ & \iddots & \\ \Gamma^{-1} & & 0 \end{bmatrix},$$

hence $P_i = 0$ for $i = 2, \ldots, \rho$. Writing $P = P_1$, it follows that $\hat{A}$ takes the form in (5.9) and $P = \mathcal{N}\tilde{A}^\rho \tilde{B}\Gamma^{-1}$.

STEP (iii): Finally we prove part (iii) of the lemma. Writing

$$M_1(s) = \begin{bmatrix} sI - \tilde{A} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix}, \quad M_2(s) = \begin{bmatrix} \mathcal{U} & 0 \\ 0 & I \end{bmatrix} M_1(s) \begin{bmatrix} \mathcal{U}^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} sI - \hat{A} & \hat{B} \\ \hat{C} & 0 \end{bmatrix},$$

and

$$M_3(s) = \begin{bmatrix} \hat{C} & 0 \\ \hat{A} - sI & -\hat{B} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & \ldots & 0 & 0 & 0 \\ -sI & I & 0 & \ldots & 0 & 0 & 0 \\ 0 & -sI & I & & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & \ldots & -sI & I & 0 & 0 \\ R_1 & R_2 & \ldots & R_{\rho-1} & R_\rho - sI & S & -\Gamma \\ P & 0 & \ldots & 0 & 0 & Q - sI & 0 \end{bmatrix},$$

we see that

$$|\det M_1(s)| = |\det M_2(s)| = |\det M_3(s)| = |\det \Gamma \, \det(sI - Q)|.$$

By the minimum-phase property of $(\tilde{A}, \tilde{B}, \tilde{C})$, we have $\det(M_1(s)) \neq 0$ for all $s \in \overline{\mathbb{C}}_+$ and so $\det(sI - Q) \neq 0$ for all $s \in \overline{\mathbb{C}}_+$. It follows that $\mathrm{spec}(Q) \subset \mathbb{C}_-$ and hence Assertion (iii) holds. □

Invoking the similarity transformation (5.8)–(5.9) and writing $x^0 := \mathcal{C}w^0$, $z^0 := \mathcal{N}w^0$, $x(t) := \mathcal{C}w(t)$, it is readily verified that system (5.7) is equivalent to

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + f(p(t), (Ty)(t), x(t)) + Bu(t), \quad x(0) = x^0, \\ y(t) &= Cx(t), \end{aligned}\right\} \tag{5.12}$$

where A, B and C are as in (5.3)–(5.4), $p\colon t \mapsto S(\exp Qt)z^0$, T is the linear operator given by

$$(Ty)(t) = S\left(\int_0^t \exp(Q(t-s))Py(s)\mathrm{d}s\right), \quad t \geq 0$$

and the function f takes the special form (5.5) with $g\colon \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ given by $g(p,w) := p + w$.

If (5.7) has sign-definite high-frequency gain, then $\tilde{C}\tilde{A}^{\rho-1}\tilde{B} = \Gamma = CA^{\rho-1}B$ is either positive definite or negative definite and hence Assumption (B1) is satisfied. If we assume that (5.7) has the minimum-phase property, then by Lemma 5.3.1 (iii), Q has spectrum in $\mathbb{C}_-$: it follows that $p \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ and T belongs to the class of operators $\mathcal{T}_0^m$ and so Assumptions (B2) and (B3) are satisfied. Assumption (B4) is trivially satisfied. Therefore, the system class Σ_ρ contains all m-input, m-output, finite-dimensional, linear, minimum-phase systems of relative degree ρ with sign-definite high-frequency gain.

5.3.2 Nonlinear systems

In [33, (1)] the following class of single-input, single-output systems is studied

$$\left.\begin{aligned} \dot{x}_1(t) &= x_2(t) + f_1(w(t), y(t)), \\ &\vdots \\ \dot{x}_{\rho-1}(t) &= x_\rho(t) + f_{\rho-1}(w(t), y(t)), \\ \dot{x}_\rho(t) &= \gamma\, u(t) + f_\rho(w(t), y(t)), \\ \dot{w}(t) &= q(w(t), y(t)), \\ y(t) &= x_1(t), \\ (x_1(0), \dots, x_\rho(0), w(0)) &= (x_1^0, \dots, x_\rho^0, w^0), \end{aligned}\right\} \tag{5.13}$$

where $\gamma \in \mathbb{R} \setminus \{0\}$, $q\colon \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$ and $f_i\colon \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, \rho$, are locally Lipschitz functions. Denote, by T, the mapping $y \mapsto w$ induced by the subsystem $\dot{w} = q(w,y)$ with initial condition $w(0) = w^0$. Then (5.13) is equivalent to (5.2) (with $h = 0$ and $m = 1$). Moreover, if we assume that the subsystem $\dot{w} = q(w,y)$ is input-to-state stable (ISS, see Section 2.2.2), then, as shown in Section 2.2.2, the operator T is of class $\mathcal{T}_0^m$, in which case system (5.13), interpreted in its equivalent form (5.2), is of class Σ_ρ.

We remark that, in [33, (1)], an assumption of *integral* input-to-state stability (iISS) is imposed on the subsystem $\dot{w} = q(w,y)$, by which it is meant that there exist functions

$\theta \in KL$ and $\gamma_1, \gamma_2 \in K_\infty$ such that, for all $(w^0, y) \in \mathbb{R}^p \times L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^m)$,

$$\gamma_1(\|w(t, w^0, y)\|) \leq \theta(\|w^0\|, t) + \int_0^t \gamma_2(\|y(s)\|)ds \quad \forall t \geq 0,$$

see, for example, [63] and [33]. The condition is strictly weaker than the assumption of ISS. In this respect, the full generality of the system class in [33] is not captured by the class Σ_ρ.

5.3.3 Delays and hysteresis

In Chapter 2, details of a variety of nonlinear effects incorporated in the operator class $\mathcal{T}_h^m$ were provided. The class of systems Σ_ρ inherits each of these nonlinearities, including delay elements and a wide range of hysteresis operators, many of which are physically motivated effects: as observed in [25], examples such as relay hysteresis, backlash hysteresis, elastic-plastic hysteresis and Preisach operators are of class $\mathcal{T}_0^1$.

5.4 The control

As in Chapter 4, a non-adaptive controller will be constructed, implementing a "back-stepping" procedure in conjunction with a filter/pre-compensator. The backstepping procedure is akin to that of [71, 33, 48].

Let $\varphi \in \Phi$ determine a performance funnel $\mathcal{F}_\varphi$. We proceed to construct a feedback structure which ensures that, for every reference $r \in \mathcal{R}$ and when applied to any system of class Σ_ρ, the tracking error $e = y - r$ evolves within $\mathcal{F}_\varphi$. As in Chapter 4, we initially assume $\rho \geq 2$; the case of systems with relative degree $\rho = 1$ will be treated separately later in this chapter.

5.4.1 Filter

Fix $\mu > 0$ and recall the filter (4.2), introduced in Chapter 4,

$$\begin{aligned}
\dot{\xi}_i(t) &= -\mu\,\xi_i(t) + \xi_{i+1}, & \xi_i(0) &= \xi_i^0 \in \mathbb{R}^m, i = 1, \ldots, \rho-2,\\
\dot{\xi}_{\rho-1}(t) &= -\mu\,\xi_{\rho-1}(t) + u(t), & \xi_{\rho-1}(0) &= \xi_{\rho-1}^0 \in \mathbb{R}^m,
\end{aligned}$$

which, on writing ξ, F and G as in (4.3), may be expressed as

$$\dot{\xi}(t) = F\xi(t) + Gu(t), \qquad \xi(0) = \xi^0 \in \mathbb{R}^{(\rho-1)m}. \tag{5.14}$$

Remark 5.4.1 Recall from Chapter 4 that the parameter $\mu \in (0,\infty)$ in the matrix F can be incorporated into the analysis with relative ease. However, a more general representation of the filter could involve arbitrary negative eigenvalues on the diagonal of F.

5.4.2 Feedback

Define

$$s(\Gamma) := \begin{cases} +1\,, & \Gamma \text{ positive definite,} \\ -1\,, & \Gamma \text{ negative definite.} \end{cases}$$

Let $\nu\colon \mathbb{R} \to \mathbb{R}$ be any C^∞ function with the property:

$$\left.\begin{array}{l}\text{there exists a strictly increasing unbounded sequence } (k_j) \text{ such that} \\ \text{the sequence } \big(s(\Gamma)\nu(k_j)\big) \text{ is strictly decreasing and unbounded.}\end{array}\right\} \tag{5.15}$$

Recall, from (4.6), the projections

$$\pi_i\colon \mathbb{R}^{(\rho-1)m} \to \mathbb{R}^{im}, \quad \xi = (\xi_1,\ldots,\xi_{\rho-1}) \mapsto (\xi_1,\ldots,\xi_i), \qquad i = 1,\ldots,\rho-1.$$

The controller implemented in this chapter is constructed in much the same way as that of Chapter 4. We repeat the recursive construction (4.7)–(4.9) for convenience, beginning with the C^∞ function

$$\gamma_1\colon \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}^m, \qquad (k,e) \mapsto \gamma_1(k,e) := -\nu(k)e\,, \tag{5.16}$$

with derivative (Jacobian matrix function) $D\gamma_1$. For $i = 2,\ldots,\rho-1$, define the C^∞ function $\gamma_i\colon \mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^{(i-1)m} \to \mathbb{R}^m$ by the recursion

$$\begin{aligned}\gamma_i(k,e,\pi_{i-1}\xi) &:= \gamma_{i-1}(k,e,\pi_{i-2}\xi) \\ &+ \|D\gamma_{i-1}(k,e,\pi_{i-2}\xi)\|^2\, k^4\,(1+\|\pi_{i-1}\xi\|^2)\Big(\mu^{2-i}\xi_{i-1} + \gamma_{i-1}(k,e,\pi_{i-2}\xi)\Big)\,,\end{aligned} \tag{5.17}$$

wherein we continue to adopt the notational convention $\gamma_1(k,e,\pi_0\xi) := \gamma_1(k,e)$. Define the C^∞ function $\gamma_\rho\colon \mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^{(\rho-1)m} \to \mathbb{R}^m$ as follows

$$\begin{aligned}\gamma_\rho(k,e,\pi_{\rho-1}\xi) &:= \mu^{\rho-1}\gamma_{\rho-1}(k,e,\pi_{\rho-2}\xi) \\ &+ \mu^{\rho-1}\|D\gamma_{\rho-1}(k,e,\pi_{\rho-2}\xi)\|^2\, k^4\,(1+\|\pi_{\rho-1}\xi\|^2)\Big(\mu^{2-\rho}\xi_{\rho-1} + \gamma_{\rho-1}(k,e,\pi_{\rho-2}\xi)\Big).\end{aligned} \tag{5.18}$$

Finally, we introduce the bijection

$$\alpha\colon [0,1) \to [1,\infty), \quad s \mapsto 1/(1-s). \tag{5.19}$$

For arbitrary $r \in \mathcal{R}$, the control strategy is given by

$$\left.\begin{aligned} u(t) &= -\gamma_\rho\big(k(t), Cx(t) - r(t), \xi(t)\big), \\ k(t) &= \alpha\big(\varphi^2(t)\|Cx(t) - r(t)\|^2\big). \end{aligned}\right\} \tag{5.20}$$

Remark 5.4.2

(i) If $s(\Gamma)$ is known *a priori*, then the function $\nu\colon k \mapsto -s(\Gamma)k$ is sufficient to ensure property (5.15); if $s(\Gamma)$ is unknown, then the function $\nu\colon k \mapsto k\cos k$ suffices. In the latter case, the role of the function ν is similar to that of a "Nussbaum" function in adaptive control.

(ii) The function α in (5.19) may be generalized to any C^∞ bijection $\alpha\colon [0,1) \to [1,\infty)$ with the property that $\alpha' = d(\alpha)$ for some function d; the particular choice $d(\cdot) = (\cdot)^2$ yields the specific function adopted in (5.19) for simplicity of presentation. In the case of general α, the term k^4 in (5.17) and (5.18) should be replaced by $d^2(k)$.

(iii) In the specific case of a system of relative degree $\rho = 2$, writing $e(t) = Cx(t) - r(t)$ and omitting the argument t for simplicity, the control strategy takes the explicit form

$$\begin{aligned} u &= \mu\,\nu(k)e - \mu\big[(\nu'(k)\|e\|)^2 + (\nu(k))^2\big]\,k^4\,[1 + \|\xi\|^2]\theta, \\ k &= \big[1 - \varphi^2\|e\|^2\big]^{-1}, \qquad \theta = \xi - \nu(k)e, \\ \dot\xi &= -\mu\,\xi + u, \qquad \xi(0) = \xi^0. \end{aligned} \tag{5.21}$$

We will make use of this controller in an example, see Section 5.7.

5.4.3 Well-posedness of the closed-loop system

The conjunction of the filter (5.14) and the feedback (5.20) applied to (5.2) yields the initial-value problem

$$\left.\begin{aligned} &\dot x(t) = A\,x(t) + f(p(t), (TCx)(t), x(t)) - B\gamma_\rho(k(t), Cx(t) - r(t), \xi(t)), \\ &\dot\xi(t) = F\xi(t) - G\gamma_\rho(k(t), Cx(t) - r(t), \xi(t)), \\ &k(t) = \alpha\big(\varphi^2(t)\|Cx(t) - r(t)\|^2\big), \\ &x|_{[-h,0]} = x^0 \in C([-h,0], \mathbb{R}^{\rho m}), \quad \xi(s) = \xi^0 \in \mathbb{R}^{(\rho-1)m} \quad \forall s \in [-h,0]. \end{aligned}\right\} \tag{5.22}$$

By a solution of (5.22) on $[-h, \omega)$ we mean a function $(x, \xi) \in C([-h, \omega), \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m})$, with $0 < \omega \leq \infty$, $x|_{[-h,0]} = x^0$ and $\xi(s) = \xi^0$ for all $s \in [-h, 0]$, such that $(x, \xi)|_{[0,\omega)}$ is absolutely continuous, satisfies the differential equations in (5.22) for almost all $t \in [0, \omega)$ and avoids the singularity in α in the sense that

$$\varphi(t)\|Cx(t) - r(t)\| < 1 \quad \forall t \in [0, \omega).$$

To answer affirmatively the question of well-posedness of the closed-loop, we make use of the existence theory constructed in Section 3.1.

Theorem 5.4.3 *Let $(A, B, C, f, p, T, h) \in \Sigma_\rho$ with $\rho \geq 1$ and let $\varphi \in \Phi$. For every $r \in \mathcal{R}$ and $(x^0, \xi^0) \in C([-h, 0], \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m})$, application of the feedback* (5.20) *in conjunction with the filter* (5.14) *to the system* (5.2) *yields the initial-value problem* (5.22) *which has a solution and every solution can be extended to a maximal solution. If a maximal solution of* (5.22) *on $[-h, \omega)$ is bounded and such that the associated gain function k is also bounded, then $\omega = \infty$.*

Proof. Introducing the open set

$$\mathcal{D} := \left\{(t, x, \xi) \in \mathbb{R}_+ \times \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m} \,\middle|\, \varphi(t)\,\|Cx - r(t)\| < 1\right\},$$

and defining, on $\mathcal{D}$,

$$\gamma_\rho^* \colon (t, x, \xi) \mapsto \gamma_\rho\big(\alpha(\varphi^2(t)\|Cx - r(t)\|^2), Cx - r(t), \xi\big),$$

the initial-value problem (5.22) may be recast on $\mathcal{D}$ as

$$\left.\begin{aligned}
&\dot{x}(t) = Ax(t) + f(p(t), T(Cx)(t), x(t)) - B\gamma_\rho^*(t, x(t), \xi(t)),\\
&\dot{\xi}(t) = F\xi(t) - G\gamma_\rho^*(t, x(t), \xi(t)),\\
&(x, \xi)|_{[-h,0]} = (x^0, \xi^0) \in C([-h, 0], \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}),\\
&(0, x^0(0), \xi^0(0)) \in \mathcal{D}.
\end{aligned}\right\} \tag{5.23}$$

Setting $\zeta = (x, \xi)$ and defining the Carathéodory function

$$Z \colon \mathcal{D} \times \mathbb{R}^q \to \mathbb{R}^{(2\rho-1)m}$$

$$(t, \zeta, w) \mapsto Z(t, \zeta, w) := \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \zeta + \begin{bmatrix} I \\ 0 \end{bmatrix} f(p(t), w, x) - \begin{bmatrix} B \\ G \end{bmatrix} \gamma_\rho^*(\zeta),$$

we can rewrite (5.23) as follows

$$\dot{\zeta}(t) = Z(t, \zeta(t), (\widehat{T}\zeta)(t)), \qquad \zeta|_{[-h,0]} = \zeta^0 \in C([-h,0], \mathbb{R}^{(2\rho-1)m}), \tag{5.24}$$

where the operator $\widehat{T}$, given by $(\widehat{T}\zeta)(t) = (TCx)(t)$, is of class $\mathcal{T}_h^{(2\rho-1)m+1}$. We then appeal directly to the existence result, Theorem 3.1.1, in Chapter 3 to conclude: (i) the existence of a solution to (5.23) and (ii) every solution can be extended to a maximal solution $(x, \xi) \in C([-h, \infty), \mathbb{R}^{(2\rho-1)m})$. Furthermore, if there exists a compact set $\mathcal{K} \subset \mathcal{D}$ such that $(t, x(t), \xi(t)) \in \mathcal{K}$ for all $t \in [0, \omega)$, then $\omega = \infty$.

Clearly, a solution $(x, \xi) \colon [-h, \omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ of (5.23) is also a solution of (5.22); conversely, a solution $(x, \xi) \colon [-h, \omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ of (5.22) is a solution of (5.23). Therefore, we may conclude that, for each $(x^0, \xi^0) \in \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$, the initial-value problem (5.22) has a solution and every solution can be maximally extended.

Let $(x, \xi) \colon [-h, \omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ be a maximal solution of (5.22). Assume that (x, ξ) is bounded and that the gain function $t \mapsto k(t) = \alpha\big(\varphi^2(t)\|Cx(t) - r(t)\|^2\big)$ is also bounded. Then there exist $c > 0$ and $\varepsilon > 0$ such that $\|(x(t), \xi(t))\| \leq c$ and $\varphi(t)\|Cx(t) - r(t)\| \leq 1 - \varepsilon$ for all $t \in [0, \omega)$. Seeking a contradiction, suppose that $\omega < \infty$. It then follows that

$$\mathcal{K} := \big\{(t, \hat{x}, \hat{\xi}) \in \mathcal{D} \big|\ \varphi(t)\|C\hat{x} - r(t)| \leq 1 - \varepsilon,\ \|(\hat{x}, \hat{\xi})\| \leq c,\ t \in [0, \omega]\big\}$$

is a compact subset of $\mathcal{D}$ such that $(t, x(t), \xi(t)) \in \mathcal{D}$ for all $t \in [0, \omega)$. This contradicts the last assertion of Theorem 3.1.1, and so $\omega = \infty$. □

5.5 Preliminary lemmas

Let $(A, B, C, f, p, T, h) \in \Sigma_\rho$ with $\rho \geq 2$. Rewriting the conjunction of the nonlinear system (5.2) and the filter (5.14) as

$$\left.\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} f(p(t), (Ty)(t), x(t)) + \begin{bmatrix} B \\ G \end{bmatrix} u(t), \\ y(t) &= [C \,\vdots\, 0] \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}, \end{aligned}\right\} \tag{5.25}$$

we have the following technicality.

Lemma 5.5.1 *For system* (5.25), *there exist* $K \in \mathbb{R}^{\rho m \times (\rho-1)m}$ *and* $N \in \mathbb{R}^{(\rho-1)m \times \rho m}$

such that

$$L := \begin{bmatrix} C & 0 \\ N & -NK \\ 0 & I \end{bmatrix} \in \mathbb{R}^{(2\rho-1)m \times (2\rho-1)m}$$

is invertible and

$$L \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} L^{-1} = \begin{bmatrix} A_1 & A_2 & \tilde{\Gamma} \\ A_3 & A_4 & 0 \\ 0 & 0 & F \end{bmatrix}, \quad L \begin{bmatrix} B \\ G \end{bmatrix} = \begin{bmatrix} 0 \\ G \end{bmatrix}, \quad [C \,\vdots\, 0]\, L^{-1} = [I \,\vdots\, 0 \,\vdots\, 0],$$

where $\tilde{\Gamma} := \left[\Gamma \,\vdots\, 0\right] \in \mathbb{R}^{m \times (\rho-1)m}$, $\Gamma := CA^{\rho-1}B$ *and* $A_4 \in \mathbb{R}^{(\rho-1)m \times (\rho-1)m}$ *is such that* $\operatorname{spec}(A_4) \subset \mathbb{C}_-$.

Proof. The proof follows immediately from Lemma 4.3.1. □

In view of Lemma 5.5.1, there exist K and N such that, under the coordinate change

$$\begin{bmatrix} y(t) \\ z(t) \\ \xi(t) \end{bmatrix} = L \begin{bmatrix} x(t) \\ \xi(t) \end{bmatrix}, \quad \begin{bmatrix} y^0 \\ z^0 \\ \xi^0 \end{bmatrix} = L \begin{bmatrix} x^0 \\ \xi^0 \end{bmatrix}, \quad L := \begin{bmatrix} C & 0 \\ N & -NK \\ 0 & I \end{bmatrix}, \tag{5.26}$$

the conjunction (5.25) of system (5.2) and filter (5.14) can be represented by

$$\left.\begin{aligned} \dot{y}(t) &= A_1 y(t) + A_2 z(t) + Cf(p(t), (Ty)(t), x(t)) + \Gamma \xi_1(t), \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t) + Nf(p(t), (Ty)(t), x(t)), \\ \dot{\xi}(t) &= F\xi(t) + Gu(t), \\ (y, z, \xi)|_{[-h,0]} &= (y^0, z^0, \xi^0) \in C([-h, 0], \mathbb{R}^m \times \mathbb{R}^{(\rho-1)m} \times \mathbb{R}^{(\rho-1)m}), \end{aligned}\right\} \tag{5.27}$$

where $A_4 \in \mathbb{R}^{(\rho-1)m \times (\rho-1)m}$ has spectrum in $\mathbb{C}_-$. If $(x, \xi)\colon [0, \omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ is a maximal solution of the nonlinear closed-loop system (5.22), then, in view of (5.27) and writing

$$y(t) = Cx(t), \qquad e(t) = y(t) - r(t), \quad e|_{[-h,0]} = e^0(\cdot) = y^0(\cdot) - r(0)\,, \tag{5.28}$$

we arrive at the following equivalent to (5.22)

$$\left.\begin{aligned}
\dot{e}(t) &= A_1 e(t) + A_2 z(t) + f_1(t) + \Gamma\xi_1(t),\\
\dot{z}(t) &= A_3 e(t) + A_4 z(t) + f_2(t),\\
\dot{\xi}(t) &= F\xi(t) - G\gamma_\rho(k(t), e(t), \xi(t)),\\
k(t) &= \alpha\big(\varphi^2(t)\|e(t)\|^2\big),\\
(e, z, \xi)|_{[-h,0]} &= (e^0, z^0, \xi^0) \in C([-h,0], \mathbb{R}^m \times \mathbb{R}^{(\rho-1)m} \times \mathbb{R}^{(\rho-1)m}),
\end{aligned}\right\} \tag{5.29}$$

where the functions f_1 and f_2 are given by

$$\left.\begin{aligned}
f_1(t) &:= A_1 r(t) + Cf(p(t), (Ty)(t), x(t)) - \dot{r}(t),\\
f_2(t) &:= A_3 r(t) + Nf(p(t), (Ty)(t), x(t)).
\end{aligned}\right\} \tag{5.30}$$

Since $(\varphi(t)\|e(t)\|)^2 < 1$ for all $t \in [0, \omega)$, the properties of $\varphi \in \Phi$ yield boundedness of the function e which, together with boundedness of r, implies boundedness of y. Since T is of class $\mathcal{T}_h^m$ and y is bounded, Ty is essentially bounded. By boundedness of r, essential boundedness of $\dot{r}$ and p, and Assumption (B4), we may now conclude (essential) boundedness of the functions f_1 and f_2. Observing that A_4 is Hurwitz and f_2 is bounded, the second of the differential equations in (5.29) yields boundedness of z. These observations are recorded in the following lemma.

Lemma 5.5.2 *Let $(A, B, C, f, p, T, h) \in \Sigma_\rho$ with $\rho \geq 2$. Let $\varphi \in \Phi$, $r \in \mathcal{R}$ and $(x^0, \xi^0) \in C([-h,0], \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m})$. If $(x, \xi)\colon [-h, \omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ is a maximal solution of* (5.22)*, then the functions y, z and e, given by* (5.26) *and* (5.28)*, are bounded. Furthermore, the functions f_1 and f_2, given by* (5.30)*, are essentially bounded and bounded, respectively.*

As in Chapter 4, the proofs of the main results (Theorems 5.6.2 and 5.6.1 below) rely on a further technicality: the signals $\theta_i = \mu^{1-i}\xi_i + \gamma_i(k, e, \pi_{i-1}\xi)$, $i = 1, ..., \rho - 1$, are bounded. More precisely, we show the following.

Lemma 5.5.3 *Let $(A, B, C, f, p, T, h) \in \Sigma_\rho$ with $\rho \geq 2$. Let $\varphi \in \Phi$, $r \in \mathcal{R}$ and $(x^0, \xi^0) \in C([-h,0], \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m})$. If $(x, \xi)\colon [-h, \omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ is a maximal solution of* (5.22)*, then the function*

$$\theta = (\theta_1, \ldots, \theta_{\rho-1})\colon [0, \omega) \to \mathbb{R}^{(\rho-1)m}$$

is bounded, where

$$\theta_i(t) := \mu^{1-i}\,\xi_i(t) + \gamma_i(k(t), e(t), \pi_{i-1}\xi(t))\,, \qquad i = 1, \ldots, \rho - 1, \tag{5.31}$$

with the notational convention $\gamma_1(k, e, \pi_0\xi) := \gamma_1(k, e)$.

Proof. Assume that $(x, \xi)\colon [-h, \omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ is a maximal solution of (5.22). Write $y(t)$ and $e(t)$ as described in (5.28). By Lemma 5.5.1, there exists an invertible linear transformation L under which the closed-loop system (5.22) may be expressed in the form (5.29), wherein, by Lemma 5.5.2, e and z are bounded and the functions f_1 and f_2 given by (5.30) are essentially bounded and bounded respectively. By boundedness of z, essential boundedness of f_1 and the first of equations (5.29), we may infer the existence of $c_1 > 0$ such that

$$\|\dot e(t)\| \le c_1\bigl(1 + \|\xi_1(t)\|\bigr) \quad \text{for a.a. } t \in [0, \omega).$$

By boundedness of φ, e, essential boundedness of $\dot\varphi$ and recalling that $\alpha'(s) = \alpha^2(s) \ge 1$ for all $s \in [0, 1)$, there exists a constant $c_2 > 0$ such that

$$\begin{aligned} |\dot k(t)| &= 2\alpha'(\varphi^2(t)\|e(t)\|^2)\bigl|\varphi^2(t)\langle e(t), \dot e(t)\rangle + \varphi(t)\dot\varphi(t)\|e(t)\|^2\bigr| \\ &\le c_2 k^2(t)\,(1 + \|\xi_1(t)\|) \quad \text{for a.a. } t \in [0, \omega). \end{aligned}$$

Noting that we have arrived at the counterpart of (4.20), boundedness of θ follows via an argument analogous to that found in the proof of Lemma 4.3.3. $\square$

5.6 Main results

5.6.1 Relative degree $\rho \ge 2$ case

We now arrive at the main result of the chapter.

Theorem 5.6.1 *Let* $(A, B, C, f, p, T, h) \in \Sigma_\rho$ *with* $\rho \ge 2$ *and let* $\varphi \in \Phi$ *with associated performance funnel* $\mathcal{F}_\varphi$. *For each reference signal* $r \in \mathcal{R}$ *and initial data* $(x^0, \xi^0) \in C([-h, 0], \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m})$, *application of the feedback* (5.20), *in conjunction with the filter* (5.14), *to* (5.2) *yields the initial-value problem* (5.22) *which has a solution and every solution can be maximally extended. Every maximal solution* $(x, \xi)\colon [-h, \omega) \to \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m}$ *has the properties:*

(i) $\omega = \infty$;

(ii) x, ξ, k *and* u *are bounded;*

(iii) *the tracking error evolves within the funnel* $\mathcal{F}_\varphi$ *and is bounded away from the funnel boundary, i.e. there exists* $\varepsilon > 0$ *such that, for all* $t \geq 0$, $\varphi(t)\|Cx(t) - r(t)\| \leq 1 - \varepsilon$.

Proof. Let (x^0, ξ^0) be arbitrary. By Theorem 5.4.3, (5.22) has a solution and every solution can be maximally extended. Let (x, ξ) be a maximal solution of (5.22) with interval of existence $[-h, \omega)$. Writing $y(t) = Cx(t)$, $e(t) = y(t) - r(t)$ for all $t \in [0, \omega)$ and invoking Lemma 5.5.1, there exists an invertible linear transformation L which takes (5.22) into the equivalent form (5.29)–(5.30). Introducing $\theta_1 \colon [0, \omega) \to \mathbb{R}^m$ given by (5.31), viz.

$$\theta_1(t) = \xi_1(t) - \nu(k(t))e(t),$$

then the first of equations (5.29) yields

$$\dot{e}(t) = f_3(t) + \nu(k(t))\,\Gamma\, e(t) \quad \text{for a.a. } t \in [0, \omega), \tag{5.32}$$

with

$$f_3(t) := A_1 e(t) + A_2 z(t) + \Gamma\theta_1(t) + f_1(t).$$

By Lemmas 5.5.2 and 5.5.3, the functions y, z, e and $\theta = (\theta_1, ..., \theta_{\rho-1})$, given by (5.31), are bounded which, together with essential boundedness of f_1, implies essential boundedness of f_3. Therefore, there exists $c_5 > 0$ such that

$$\langle e(t), \dot{e}(t)\rangle \leq c_5 + \nu(k(t))\,\langle e(t), \Gamma e(t)\rangle \quad \text{for a.a. } t \in [0, \omega). \tag{5.33}$$

We are now in a position to prove boundedness of k. Recalling that Γ is either positive definite or negative definite, there exist constants $\beta_0, \beta_1 > 0$ such that

$$\beta_0\|e\|^2 \leq |\langle e, \Gamma e\rangle| \leq \beta_1\|e\|^2 \quad \forall e \in \mathbb{R}^m.$$

Define the continuous function $\tilde{\nu} \colon \mathbb{R} \to \mathbb{R}$ as follows

$$\tilde{\nu}(k) := \begin{cases} -\beta_1\nu(k), & s(\Gamma)\nu(k) \geq 0\,, \\ -\beta_0\nu(k), & s(\Gamma)\nu(k) < 0\,. \end{cases}$$

Observe that

$$\nu(k)\langle e, \Gamma e\rangle \leq -s(\Gamma)\tilde{\nu}(k)\|e\|^2 \quad \forall e \in \mathbb{R}^m \quad \forall k \geq 0\,,$$

which, together with boundedness of e, φ, essential boundedness of $\dot{\varphi}$ and (5.33), implies the existence of $c_6 > 0$ such that

$$\begin{aligned}\frac{d}{dt}(\varphi(t)\|e(t)\|)^2 &= 2\varphi(t)\dot{\varphi}(t)\|e(t)\|^2 + 2\varphi^2(t)\langle e(t), \dot{e}(t)\rangle \\ &\le c_6 - 2s(\Gamma)\,\tilde{\nu}(k(t))\,\big(\varphi(t)\|e(t)\|\big)^2 \quad \text{for a.a. } t \in [0,\omega).\end{aligned}$$

In view of property (5.15) of ν, there exists a strictly increasing unbounded sequence (k_j) in $(1,\infty)$ such that the sequence $\big(s(\Gamma)\tilde{\nu}(k_j)\big)$ is also strictly increasing, unbounded and such that $s(\Gamma)\tilde{\nu}(k_j) > 0$ for all $j \in \mathbb{N}$. Seeking a contradiction, suppose k is unbounded on $[0,\omega)$. For each $j \in \mathbb{N}$, define

$$\begin{aligned}\tau_j &:= \inf\{t \in [0,\omega) \mid k(t) = k_{j+1}\}, \\ \sigma_j &:= \sup\{t \in [0,\tau_j] \mid \tilde{\nu}(k(t)) = \tilde{\nu}(k_j)\}.\end{aligned}$$

It can easily be verified that $\sigma_j < \tau_j$ and $k(\sigma_j) < k(\tau_j)$; moreover, for all $j \in \mathbb{N}$ and all $t \in [\sigma_j, \tau_j]$, $k(t) \ge k_j$ and $s(\Gamma)\tilde{\nu}(k(t)) \ge s(\Gamma)\tilde{\nu}(k_j)$. Therefore,

$$(\varphi(t)\|e(t)\|)^2 \ge \alpha^{-1}(k_j) \ge \alpha^{-1}(k_1) = 1 - \frac{1}{k_1} =: c_7 > 0 \quad \forall t \in [\sigma_j, \tau_j] \quad \forall\, j \in \mathbb{N},$$

where $\alpha^{-1}\colon [1,\infty) \to [0,1)$ is the inverse of the bijection α. Thus,

$$\frac{d}{dt}(\varphi(t)\|e(t)\|)^2 \le c_6 - 2c_7 s(\Gamma)\tilde{\nu}(k(t)) \quad \forall t \in [\sigma_j, \tau_j] \quad \forall\, j \in \mathbb{N}.$$

Let $j^* \in \mathbb{N}$ be sufficiently large so that $c_6 - 2c_7 s(\Gamma)\tilde{\nu}(k_{j^*}) < 0$. Then,

$$\big(\varphi(\tau_{j^*})\|e(\tau_{j^*})\|\big)^2 < \big(\varphi(\sigma_{j^*})\|e(\sigma_{j^*})\|\big)^2,$$

whence the contradiction

$$0 > \alpha\big(\varphi^2(\tau_{j^*})\|e(\tau_{j^*})\|^2\big) - \alpha\big(\varphi^2(\sigma_{j^*})\|e(\sigma_{j^*})\|^2\big) = k(\tau_{j^*}) - k(\sigma_{j^*}) > 0.$$

This proves boundedness of k. Therefore, there exists $\varepsilon > 0$ such that $\varphi(t)\|e(t)\| \le 1-\varepsilon$ for all $t \in [0,\omega)$. By boundedness of θ, e and k, together with continuity of the functions γ_i, it follows from the recursive construction in (5.31) that, for $i = 1, ..., \rho - 1$, ξ_i is bounded. We may now deduce that x and ξ are bounded and, by (5.16), (5.17), (5.18) and (5.20), we may also infer boundedness of u. Finally, by boundedness of x, ξ and k, together with Theorem 5.4.3, we conclude that $\omega = \infty$. □

5.6.2 Relative degree 1 case

For the case of sign-definite CB (of unknown sign), in which case the system has relative degree 1, a filter is not necessary and the controller (5.20) simplifies to

$$\left.\begin{aligned} u(t) &= \nu(k(t))(Cx(t) - r(t)), \\ k(t) &= \alpha\big(\varphi^2(t)\|Cx(t) - r(t)\|^2\big)\,. \end{aligned}\right\} \tag{5.34}$$

The closed-loop initial-value problem then becomes

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + B\nu(k(t))(Cx(t) - r(t)) + f(p(t), T(Cx)(t), x(t)), \\ k(t) &= \alpha\big(\varphi^2(t)\|Cx(t) - r(t)\|^2\big)\,, \\ x|_{[-h,0]} &= x^0 \in C([-h,0], \mathbb{R}^m). \end{aligned}\right\} \tag{5.35}$$

Theorem 5.6.2 *Let $(A, B, C, f, p, T, h) \in \Sigma_1$ and $\varphi \in \Phi$ with associated performance funnel $\mathcal{F}_\varphi$. For each reference signal $r \in \mathcal{R}$, and initial data $(x^0, \xi^0) \in C([-h,0], \mathbb{R}^{\rho m} \times \mathbb{R}^{(\rho-1)m})$, application of the feedback (5.34) to (5.2) yields the initial-value problem (5.35) which has a solution and every solution can be maximally extended. Every maximal solution $x\colon [-h,\omega) \to \mathbb{R}^m$ has the properties:*

(i) *$\omega = \infty$;*

(ii) *x, k and u are bounded;*

(iii) *the tracking error evolves within the funnel $\mathcal{F}_\varphi$ and is bounded away from the funnel boundary, i.e. there exists $\varepsilon > 0$ such that, for all $t \geq 0$, $\varphi(t)\|Cx(t) - r(t)\| \leq 1 - \varepsilon$.*

Proof. The proof of Theorem 5.6.2 follows easily by modifying (all vestiges of the filter equations are excised) the proof of Theorem 5.6.1. □

Remark 5.6.3 In the context of linear systems (see the prototype in Section 5.3.1) with sign-definite CB of known sign, the counterpart of the above result was proved in [26]; the general case wherein CB is of unknown sign was then considered in Chapter 4. The nonlinear nature of systems in the class Σ_ρ ensures that the above result constitutes a generalization of the results in [26] and Chapter 4.

5.7 Examples

To allow for comparison with the simulations in Chapter 4, we illustrate the controller strategy (5.20) applied to the following single-input, single-output, system of relative degree two:

$$\ddot{y}(t) + b_0 \sin y(t) + b_1 y(t)|y(t)| + (\mathcal{B}_{a,b}(y))(t) = b_2\, u(t)\,, \tag{5.36}$$

where b_0, b_1 and $b_2 \neq 0$ are unknown real parameters and $\mathcal{B}_{a,b}$ represents a backlash operator, as defined in Section 2.2, with parameters $a > 0$ and $b \in [-a, a]$. In the absence of the nonlinearities, $y|y|$ and $\mathcal{B}_{a,b}$, the equation above is the same as the example in Chapter 4. Equation (5.36) is equivalent to (5.2) with

$$x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \vdots 0 \end{bmatrix}, \quad f(p, w, x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} w$$

and the operator T given by $(Ty)(t) = b_0 \sin y(t) + b_1 y(t)|y(t)| + (\mathcal{B}_{a,b}(y))(t)$, $t \in \mathbb{R}_+$. Setting $h = 0$ and $p = 0$, the resulting system $(A, B, C, f, 0, T, 0)$ is of class Σ_2.

Let $\lambda > 0$. Fix $\tau > 0$ arbitrarily and define $\varphi \in \Phi$ by

$$t \mapsto \varphi(t) = \begin{cases} \frac{1}{\lambda}(1 - (\frac{t}{\tau} - 1)^2), & 0 \leq t < \tau, \\ \frac{1}{\lambda}, & t \geq \tau. \end{cases} \tag{5.37}$$

Evolution within the associated performance funnel $\mathcal{F}_\varphi$ ensures a tracking accuracy $|e(t)| < \lambda$ for all $t \geq \tau$. To accommodate the unknown sign of b_2, choose $\nu\colon k \mapsto k \cos k$, then, choosing $\xi^0 = 0$, writing $e(t) = y(t) - r(t)$ and suppressing the argument t for simplicity, the control strategy (5.21) is

$$\left.\begin{array}{l} u = \mu(k \cos k)e - \mu\big[(\cos k - k \sin k)^2\, e^2 + k^2 \cos^2 k\big]\, k^4\, [1 + \xi^2]\theta, \\ k = \big[1 - \varphi^2 e^2\big]^{-1}, \qquad \theta = \xi - (k \cos k)e, \\ \dot{\xi} = -\mu\, \xi + u, \qquad \xi(0) = 0. \end{array}\right\} \tag{5.38}$$

Example (i) For comparison with Chapter 4, let $\lambda = 0.1$ and $\tau = 10$, so that the prescribed tracking accuracy is again $e(t) < 0.1$ for all $t \geq 10$. Setting $b_0 = \frac{1}{2}$, $b_1 = 1 = b_2$, $\mu = 1$ and adopting backlash hysteresis parameters $a = \frac{1}{2}$, $b = 0$, initial data $(y(0), \dot{y}(0)) = (0, 0)$ and reference signal $t \mapsto r(t) = \frac{1}{2} \cos t$, the behaviour of the closed-loop system (5.36)–(5.38) over the time interval $[0, 20]$ is depicted in Figure 5-3. The "peaks" in the control action occur whenever the tracking error is close to the boundary of the funnel. However, if $b \neq 0$ is known *a priori* to be positive, then the peaking behaviour is considerably mollified by choosing the function $\nu\colon k \mapsto -k$ in place of $k \mapsto k\cos k$, in which case the strategy is

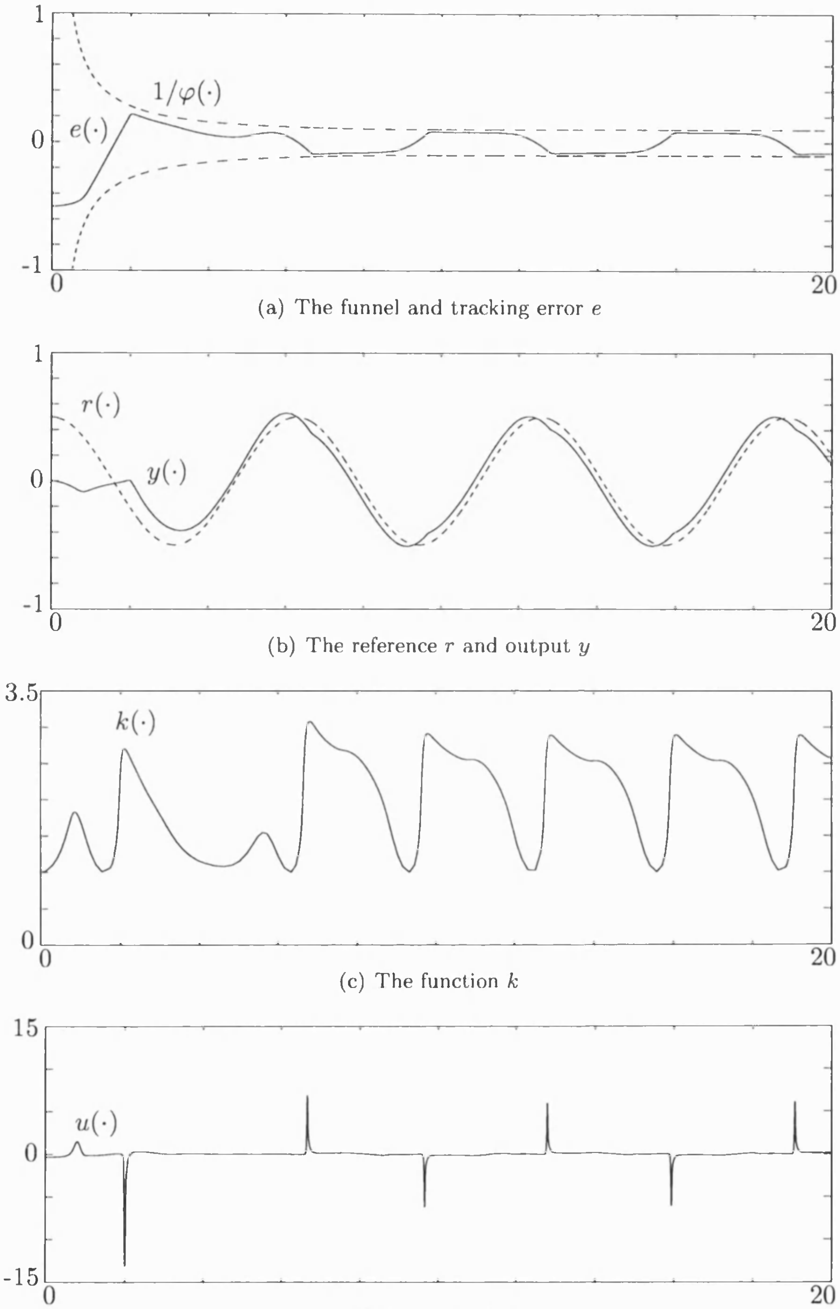
$$\left.\begin{aligned} u &= -ke - [\xi + ke]\left[e^2 + k^2\right] k^4 \left[1 + \xi^2\right], \\ k &= \left[1 - \varphi^2 e^2\right]^{-1}, \\ \dot{\xi} &= -\xi + u, \qquad\qquad \xi(0) = 0. \end{aligned}\right\} \tag{5.39}$$

For the same parameter values and initial data as above, the behaviour (5.36), under control (5.39), is shown in Figure 5-4.

Example (ii) To illustrate the diversity offered by the reference signal class $\mathcal{R}$, we take the reference to be tracked, $r \in \mathcal{R}$, to be the first component, ζ_1, of the solution of the following Lorenz system of equations:

$$\left.\begin{aligned} \dot{x}(t) &= y(t) - x(t), & x(0) &= 1, \\ \dot{y}(t) &= \tfrac{28}{10}x(t) - \tfrac{1}{10}y(t) - x(t)z(t), & y(0) &= 0, \\ \dot{z}(t) &= x(t)y(t) - \tfrac{8}{30}z(t), & z(0) &= 3. \end{aligned}\right\} \tag{5.40}$$

It is well known that the unique global solution of (5.40) is bounded with bounded derivative, see for example [65]. Let $\lambda = 0.05$ and $\tau = 25$, so prescribed tracking accuracy $|e(t)| < 0.05$ is achieved for all $t \geq 25$. Setting the filter coefficient to be $\mu = 10$, whilst maintaining the values $b_0 = \frac{1}{2}$, $b_1 = 1 = b_2$, backlash hysteresis parameters $a = \frac{1}{2}$, $b = 0$ and initial data $(y(0), \dot{y}(0)) = (0, 0)$ from Example (i), the behaviour of the closed-loop system (5.36)–(5.38) over the time interval $[0, 50]$ is depicted in Figure 5-5.



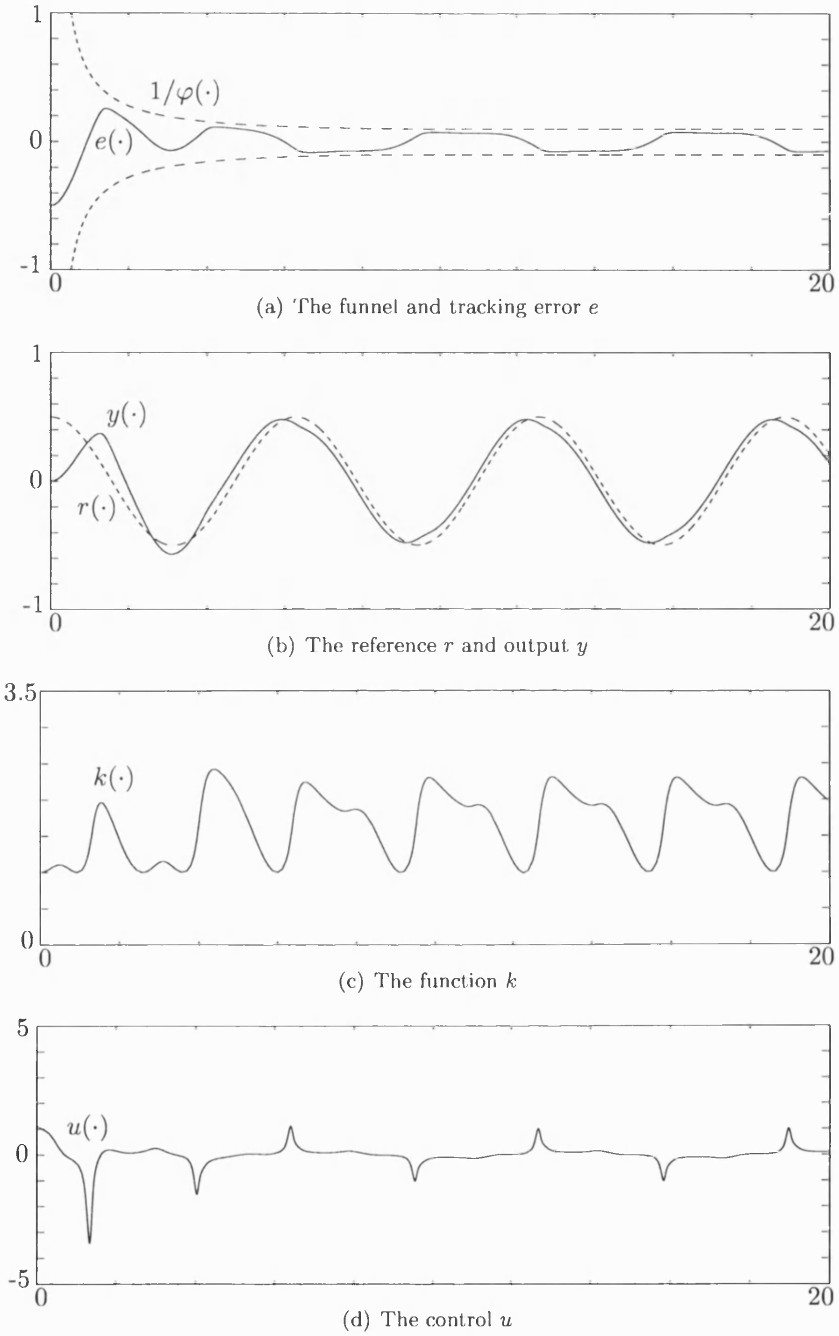
(a) The funnel and tracking error e

(b) The reference r and output y

(c) The function k

(d) The control u

Figure 5-3: Unknown sign $b_2 \neq 0$: control (5.38) applied to the nonlinear system (5.36).



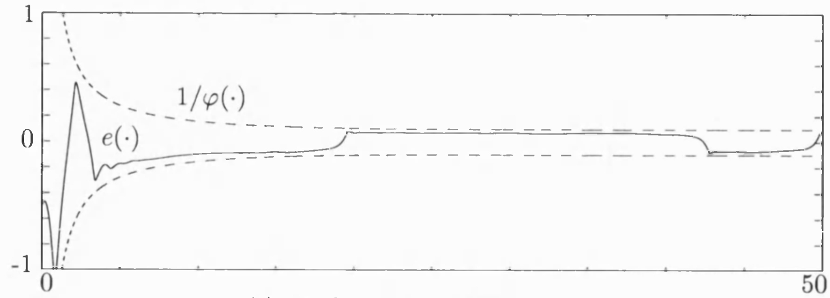
(a) The funnel and tracking error e

(b) The reference r and output y

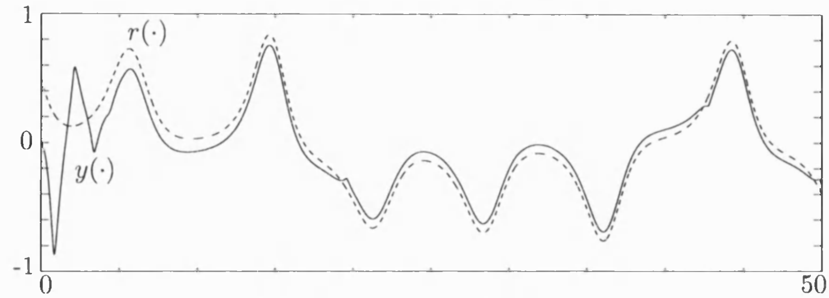
(c) The function k

(d) The control u

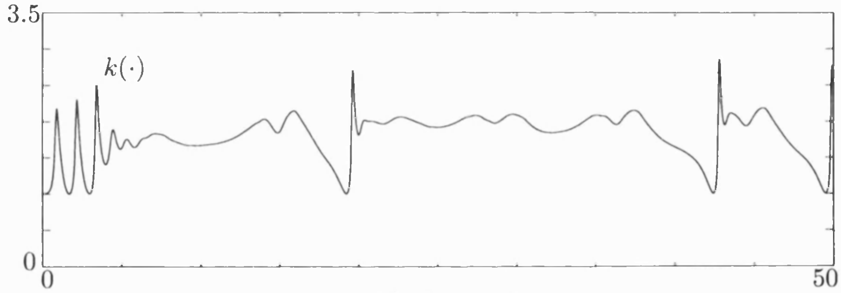
Figure 5-4: Known sign $b_2 > 0$: control (5.39) applied to the nonlinear system (5.36).



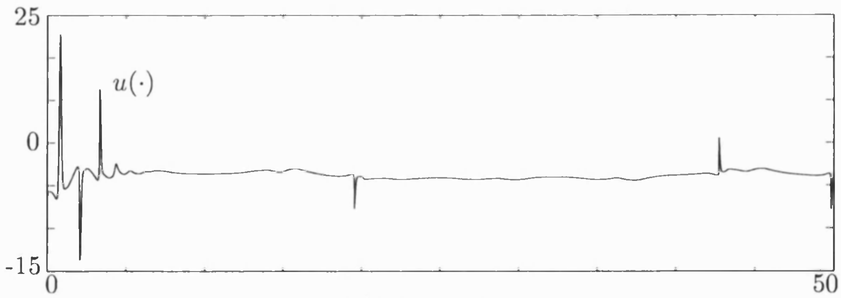
(a) The funnel and tracking error e



(b) The reference r and output y



(c) The function k



(d) The control u

Figure 5-5: Tracking of a Lorenz component reference signal; system (5.36) with unknown sign $b_2 \neq 0$ and control strategy (5.39).

Chapter 6

Asymptotic tracking and transient behaviour

A tracking problem is considered in this chapter in the context of a class of multi-input, multi-output, nonlinear systems modelled by functional differential equations. The key feature that distinguishes the current chapter (and Chapter 7 to follow) from Chapters 4 and 5 is the consideration of asymptotic tracking in addition to approximate tracking.

6.1 Introduction

In the precursors [26], Chapter 4 and Chapter 5 to the present chapter, an *approximate* tracking problem is addressed for various classes of systems. Let $\mathcal{S}$ be some given system class and let $\mathcal{R}$ be a class of reference signals. As described in Chapter 1, by approximate tracking, we mean attainment of the following: for any prescribed $\lambda > 0$, determine a *continuous* output (y) feedback strategy which ensures that, for every system in $\mathcal{S}$ and every reference signal $r \in \mathcal{R}$, (i) the tracking error $e = y - r$ is ultimately contained in the ball of radius λ centred at 0 (equivalently, $\limsup_{t\to\infty} \|e(t)\| < \lambda$), and (ii) the tracking error exhibits prescribed transient behaviour (that is, for some suitable prescribed function φ with $0 < \liminf_{t\to\infty} \varphi(t) < \infty$, we have $\|e(t)\| < 1/\varphi(t)$ for all $t > 0$).

The results in this chapter encompass not only approximate tracking but also the problem of *asymptotic* tracking with prescribed transient behaviour: in this case, an output feedback strategy (possibly discontinuous) is sought which ensures that, for every system of class $\mathcal{S}$, every reference signal $r \in \mathcal{R}$ and some suitable prescribed function φ, with $\varphi(t) \to \infty$ as $t \to \infty$, we have $\|e(t)\| < 1/\varphi(t)$ for all $t > 0$ (and so

$e(t) \to 0$ as $t \to \infty$). Both cases (approximate and asymptotic tracking) are analysed within a unified framework of functional differential inclusions.

The focus of this chapter will be nonlinear systems (akin to those considered in [26]), with control input u, modelled by functional differential equations of the form

$$\dot{y}(t) = f(d(t), (Ty)(t), u(t)), \quad y|_{[-h,0]} = y^0 \in C([-h, 0], \mathbb{R}^m), \tag{6.1}$$

where f is continuous, T is a causal operator, d may be thought of as a continuous and bounded disturbance, and $h \geq 0$ quantifies the "memory" of the system. As in [26, 27, 28], the class $\mathcal{R}$ of reference signals is taken to be the space $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$.

This chapter is structured as follows. Section 6.2 formulates the control objectives and, in Section 6.3, a full description of the system class $\mathcal{S}$ is provided. Section 6.5 details the feedback structure, the potentially discontinuous nature of which leads to an interpretation of the closed-loop system in the form of a functional differential inclusion. The existence theory developed in Chapter 3 for functional differential inclusions will be applied to the closed-loop system in Section 6.5.3. The main results of this Chapter on transient behaviour and asymptotic tracking for the closed-loop system are given in Section 6.6. This Chapter is based on joint work in [58].

6.2 Control objectives and the performance funnel

To accommodate asymptotic tracking, the control aims in this chapter differ from those introduced in Chapters 4 and 5. The two control objectives are:

(i) tracking of any reference signal $r \in \mathcal{R} := W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ by the output y. For arbitrary $\lambda \geq 0$, we seek an output feedback strategy which ensures that, for every $r \in \mathcal{R}$, every solution of the closed-loop system is bounded and the tracking error $e = y - r$ is such that $\limsup_{t\to\infty} \|e(t)\| < \lambda$ if $\lambda > 0$ or $\lim_{t\to\infty} \|e(t)\| = 0$ if $\lambda = 0$;

(ii) prescribed transient behaviour of the tracking error.

As before, both objectives are captured in the concept of a performance funnel $\mathcal{F}_\varphi$ associated, in this chapter, with a function φ belonging to Φ_λ, viz.

$$\Phi_\lambda = \big\{\varphi \in AC_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}) \;\big|\; \varphi(0) = 0,\ \varphi(s) > 0\ \ \forall s > 0,\ \liminf_{s\to\infty} \varphi(s) = 1/\lambda,$$
$$\exists\, c > 0 : \ \ \dot{\varphi}(s) \leq c[1 + \varphi(s)] \text{ for a.a. } s > 0\big\},$$

with the convention that, if $\lambda = 0$, then $1/\lambda := \infty$ (and so $\varphi(t) \to \infty$ as $t \to \infty$).

If a feedback structure can be devised which ensures that, for every system of the underlying class and every $r \in \mathcal{R}$, the graph of the tracking error $e = y - r$ is properly contained in $\mathcal{F}_\varphi$ then: (i) $\|e(t)\| < 1/\varphi(t)$ for all $t \in \mathbb{R}_+$; (ii) transient behaviour is determined by the choice of φ. A critical point to note in this chapter is that the case $\lambda = 0$ implies that $\varphi(t) \to \infty$ as $t \to \infty$, thereby ensuring the exact asymptotic tracking objective.

The intuition underpinning the feedback structure proposed in Section 6.5 matches that of the previous chapters, though a filter is not required. An intrinsic high-gain property of the system class ensures that, if $(t, e(t))$ approaches the funnel boundary, then the control input attains values sufficiently large to preclude boundary contact.

6.3 Class of systems

Fix $m \in \mathbb{N}$ arbitrarily. We now define the system class.

Definition 6.3.1 (System class $\mathcal{S}$)
The class $\mathcal{S}$ is comprised of multi-input ($u(t) \in \mathbb{R}^m$), multi-output ($y(t) \in \mathbb{R}^m$), nonlinear systems (f, d, T, h) of the form (6.1), satisfying the following assumptions.

(S1) *The function $f\colon \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous.*

(S2) *For each compact set $\mathcal{K} \subset \mathbb{R}^p \times \mathbb{R}^q$, the continuous function $\gamma_\mathcal{K}\colon \mathbb{R} \to \mathbb{R}$, given by*

$$\gamma_\mathcal{K}(s) := \min\left\{\langle v, f(l, w, sv)\rangle \mid (l, w) \in \mathcal{K}, \quad \|v\| = 1\right\}, \tag{6.2}$$

is such that either (i) $\limsup_{s\to\infty} \gamma_\mathcal{K}(s) = \infty$, or (ii) $\limsup_{s\to-\infty} \gamma_\mathcal{K}(s) = \infty$.

(S3) *$d \in C(\mathbb{R}_+, \mathbb{R}^p)$ is bounded.*

(S4) *$T\colon C([-h, \infty), \mathbb{R}^m) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^q)$ is of class $\mathcal{T}_h^m$ (see the definition and discussion in Chapter 2).*

6.4 Subclasses of $\mathcal{S}$

We highlight several subclasses of $\mathcal{S}$.

6.4.1 Linear prototype

A system (6.1) of class $\mathcal{S}$, like (5.2) in Chapter 5 (see Remark 5.2.2), can be thought of as an interconnection of two (sub) systems: the dynamical system Λ_1 and the system Λ_2, formulated as a causal operator mapping the system output y to w, see Figure 6-1.

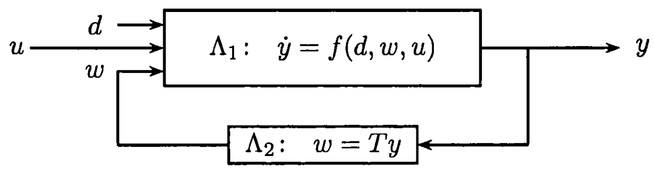


Figure 6-1: System of class $\mathcal{S}$.

To illustrate this more clearly, consider the prototype class $\mathcal{L}$ of finite-dimensional, minimum-phase, multi-input ($u(t)$), multi-output ($y(t)$) linear systems (A, B, C) with sign-definite high-frequency gain.

It was shown in Section 2.2.1 that, following an appropriate similarity transform, every system described by $(A, B, C) \in \mathcal{L}$ could be rewritten in the form

$$\dot{y}(t) = d(t) + (Ty)(t) + CB\,u(t), \quad y(0) = y^0 \in \mathbb{R}^m, \tag{6.3}$$

where the function d and operator T were given by

$$\left.\begin{array}{rcl} d(t) & := & A_2\big(\exp(A_4 t)\big)z^0 \\ (Ty)(t) & := & A_1 y(t) + A_2 \int_0^t \big(\exp A_4(t-s)\big) A_3 y(s) ds. \end{array}\right\} \tag{6.4}$$

Clearly, (6.3) is of the form (6.1) with $h = 0$ and $f\colon \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $(l, w, v) \mapsto l + w + CB\,v$. Evidently, Assumption (S1) holds. Recalling that A_4 is Hurwitz, we see that (S3) and (S4) (with $h = 0$) are valid. It remains to show that (S2) also holds. Recall that CB is sign definite and so either (i) $CB > 0$, or (ii) $-CB > 0$. Let $\mathcal{K} \subset \mathbb{R}^m \times \mathbb{R}^m$ be compact and define

$$c_{\mathcal{K}} := \min\{\langle v, l + w\rangle \mid (l, w) \in \mathcal{K},\ \|v\| = 1\}.$$

Now, observe that

$$\begin{array}{rcl} CB > 0 & \Longrightarrow & \min\{\langle v, CBv\rangle \mid \ \|v\| = 1\} = \frac{1}{2}\|(CB + (CB)^T)^{-1}\|^{-1} \\ -CB > 0 & \Longrightarrow & \min\{\langle v, CBv\rangle \mid \ \|v\| = 1\} = -\frac{1}{2}\|CB + (CB)^T\| \end{array}$$

Therefore,

(i) $CB > 0,\ s \geq 0 \implies \gamma_{\mathcal{K}}(s) \geq c_{\mathcal{K}} + \frac{1}{2}s\|(CB + (CB)^T)^{-1}\|^{-1}$, so (S2)(i) holds,

(ii) $-CB > 0,\ s \leq 0 \implies \gamma_{\mathcal{K}}(s) \geq c_{\mathcal{K}} - \frac{1}{2}s\|CB + (CB)^T\|$, so (S2)(ii) holds.

Systems with input nonlinearity

To illustrate the generality afforded by Assumption (S2), consider a single-input, single-output ($m = 1$) system described by $(A, B, C) \in \mathcal{L}$ with a nonlinearity g in the input channel

$$\left.\begin{aligned} \dot{y}(t) &= A_1 y(t) + A_2 z(t) + \beta\, g(u(t)), & y(0) &= y^0, \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t), & z(0) &= z^0, \end{aligned}\right\} \tag{6.5}$$

where $\beta := CB$ is now a non-zero real number. We assume only that $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous unbounded function with bounded even part, for example, $g\colon v \mapsto (1 + v)\cos v$. Such a function can influence/reverse the polarity of an input signal $u(\cdot)$ in a manner unpredictable by a controller. Defining d and T as in (6.4), system (6.5) can be expressed as

$$\dot{y}(t) = d(t) + (Ty)(t) + \beta g(u(t)), \quad y(0) = y^0 \in \mathbb{R},$$

which again is of form (6.1). Assumptions (S1), (S3) and (S4) clearly hold. Define g_o and g_e to be the odd and even parts, respectively, of the function βg. To see that (S2) holds, let $\mathcal{K} \subset \mathbb{R} \times \mathbb{R}$ be compact, define $c_{\mathcal{K}}$ as above, and observe that, since $v g_o(sv) = g_o(s)$ for all $|v| = 1$ and all $s \in \mathbb{R}$, we have

$$\begin{aligned} \gamma_{\mathcal{K}}(s) &= \min\{v(l + w + g_e(sv)) \mid (l, w) \in \mathcal{K},\ |v| = 1\} + g_o(s) \\ &\geq c_{\mathcal{K}} - |g_e(s)| + g_o(s) \quad \forall s. \end{aligned} \tag{6.6}$$

Since the function g_o is odd and unbounded, there must exist an unbounded monotone sequence (s_n) (either strictly increasing or strictly decreasing) such that $g_o(s_n) \to \infty$ as $n \to \infty$ which, together with boundedness of g_e and (6.6), ensures $\gamma_{\mathcal{K}}(s_n) \to \infty$ as $n \to \infty$.

6.4.2 Nonlinear systems

Now consider a further generalization of systems of form (6.5) to nonlinear systems of the form

$$\left.\begin{aligned} \dot{y}(t) &= f_1(y(t), z(t)) + g(u(t)), & y(0) &= y^0 \in \mathbb{R}, \\ \dot{z}(t) &= f_2(y(t), z(t)), & z(0) &= z^0 \in \mathbb{R}^p, \end{aligned}\right\} \tag{6.7}$$

with f_1 continuous, f_2 locally Lipschitz, and (as above) g continuous and unbounded with bounded even part (here, the parameter β is absorbed by g). Temporarily regarding y as an independent input to the second subsystem in (6.7), denote the unique solution of the initial-value problem $\dot{z} = f_2(y, z)$, $z(0) = z^0$, by $z(\cdot, z^0, y)$. If we now assume that the second subsystem in (6.7) is input-to-state stable (ISS) (recall the commentary in Section 2.2.2 or see [60]), then, for each $z^0 \in \mathbb{R}^p$, we may define an operator $C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R} \times \mathbb{R}^p)$ by

$$(Ty)(t) := (y(t), z(t, z^0, y)) \quad \forall\, t \in \mathbb{R}_+.$$

This operator T is of class $\mathcal{T}_0^1$ (Assumption (S4) holds with $h = 0$, $m = 1$ and $q = p+1$). System (6.7) may be expressed as functional differential equation

$$\dot{y}(t) = f_1((Ty)(t)) + g(u(t)), \quad y(0) = y^0,$$

which is of the form (6.1) with $h = 0$ and $f\colon (x, w, v) \mapsto f_1(w) + g(v)$. Evidently, Assumption (S1) holds, Assumption (S3) is vacuous, and Assumption (S2) holds by the same argument used in Section 6.4.1.

Remark 6.4.1 Recall that the class of operators $\mathcal{T}_h^m$ also accommodates a wide range of nonlinear effects such as delays and hysteresis; see Chapter 2 for details and also [25], [4], [40] for more background information.

6.5 Feedback control

We proceed to make precise the proposed output feedback structure. Let $\lambda \geq 0$ and $\varphi \in \Phi_\lambda$. Let $\nu\colon \mathbb{R} \to \mathbb{R}$ be any continuous function with the properties

$$\limsup_{k \to \infty} \nu(k) = +\infty \quad \text{and} \quad \liminf_{k \to \infty} \nu(k) = -\infty\,, \tag{6.8}$$

for example, $\nu\colon k \mapsto k \cos k$. Let $\alpha\colon [0, 1) \to \mathbb{R}_+$ be a continuous, unbounded injection, for example, $\alpha\colon s \mapsto s/(1-s)$. Define

$$\mu := \begin{cases} \dfrac{1}{2\sup_{t \in \mathbb{R}_+} \varphi(t)}\,, & \text{if } \varphi \text{ is bounded,} \\ 0, & \text{otherwise.} \end{cases}$$

If $\mu > 0$, let $\mathrm{sat}_\mu\colon \mathbb{R}^m \to \mathcal{B} := \{v \in \mathbb{R}^m |\ \|v\| \leq 1\}$ be any continuous function with the property that $\mathrm{sat}_\mu(e) = \|e\|^{-1} e$ for all $\|e\| > \mu$, in which case the control strategy takes the form

$$\left.\begin{aligned} u(t) &= -\nu(k(t))\mathrm{sat}_\mu(y(t)-r(t)), \\ k(t) &= \alpha(\varphi(t)\|y(t)-r(t)\|). \end{aligned}\right\}$$

In the case $\mu = 0$, the control strategy is given formally by

$$\left.\begin{aligned} u(t) &= -\nu(k(t))\|y(t)-r(t)\|^{-1}(y(t)-r(t)), \\ k(t) &= \alpha(\varphi(t)\|y(t)-r(t)\|). \end{aligned}\right\} \tag{6.9}$$

We accommodate each case and the (potential) discontinuity in (6.9) by embedding the control in a set-valued map θ_μ, defined as follows:

$$\theta_\mu(e) = \begin{cases} \{e\|e\|^{-1}\}, & \text{if } \|e\| > \mu, \\ \mathcal{B}, & \text{if } \|e\| \leq \mu \end{cases}$$

and interpret both control strategies in the following unified, set-valued sense:

$$\left.\begin{aligned} u(t) &\in -\nu(k(t))\theta_\mu(y(t)-r(t)), \\ k(t) &= \alpha(\varphi(t)\|y(t)-r(t)\|). \end{aligned}\right\} \tag{6.10}$$

If, for a given linear system (A, B, C) of prototype class $\mathcal{L}$, the polarity $\mathrm{sgn}(CB)$ of the sign-definite high-frequency gain is known *a priori*, then the term $\nu(k(t))$ in (6.10) can be replaced by $k(t)\mathrm{sgn}(CB)$.

6.5.1 Closed-loop system

Let $\lambda \geq 0$, $\varphi \in \Phi_\lambda$, $r \in \mathcal{R}$ and let $\mathcal{D} \subset \mathbb{R}_+ \times \mathbb{R}^m$ denote the set

$$\{(t,\zeta) \in \mathbb{R}_+ \times \mathbb{R}^m \mid \varphi(t)\|\zeta - r(t)\| < 1\}. \tag{6.11}$$

Let $(f, d, T, h) \in \mathcal{S}$. The conjunction of (6.1) with (6.10) yields the following closed-loop initial-value problem

$$\dot{y}(t) \in F(t, y(t), (Ty)(t)), \qquad y|_{[-h,0]} = y^0 \in C([-h,0], \mathbb{R}^m), \tag{6.12}$$

where the set-valued map $(t, y, w) \mapsto F(t, y, w) \subset \mathbb{R}^m$, given by

$$F(t,y,w) := \{f(d(t), w, u) \mid u \in -\nu\big(\alpha(\varphi(t)\|y-r(t)\|)\big)\theta_\mu(y-r(t))\}, \tag{6.13}$$

is upper semicontinuous on $\mathcal{D} \times \mathbb{R}^q$ with non-empty, convex, compact values. By a *solution* of (6.12) we mean a function $y \in C(I, \mathbb{R}^m)$ on some interval I of the form $[-h, \rho]$, $0 < \rho < \infty$ or $[-h, \omega)$, $0 < \omega \leq \infty$, such that $y|_{[-h,0]} = y^0$, $y|_J$ is locally absolutely continuous, with $(t, y(t)) \in \mathcal{D}$ for all $t \in J$ and $\dot{y}(t) \in F(t, y(t), (Ty)(t))$ for almost all $t \in J$, where $J := I \backslash [-h, 0)$. We will demonstrate that the control objectives are achieved by establishing the following facts: (i) the initial-value problem (6.12) has a solution; (ii) every solution can be extended to a maximal solution; (iii) every maximal solution is global. In Theorem 6.5.1, it will be shown that facts (i) and (ii) hold and the proof will rely on the existence theory (Theorem 3.2.1) developed in Section 3.2; fact (iii) is the essence of the main result in Theorem 6.6.1. Before proceeding to establish these facts, some commentary on the case $\lambda = 0$ is warranted.

6.5.2 Commentary on the asymptotic tracking problem

Assume $\lambda = 0$, in which case we have $\mu = 0$, and so the associated formal control structure (6.9) is potentially discontinuous. However, we remark that this need not always be the case. For example, with the choices

$$\nu\colon k \mapsto k\cos(ck) \quad \text{and} \quad \alpha\colon s \mapsto \frac{s}{1-s},$$

where $c > 0$, the feedback (6.9) is, in fact, continuous on the domain $\mathcal{D}$, viz.

$$u(t) = \psi(t, y(t) - r(t)), \tag{6.14}$$

with $\psi \in C(\mathcal{D}, \mathbb{R}^m)$ given by

$$\psi(t, \zeta) := -\cos\left(\frac{c\varphi(t)\|\zeta\|}{1 - \varphi(t)\|\zeta\|}\right)\left(\frac{\varphi(t)\zeta}{1 - \varphi(t)\|\zeta\|}\right) \quad \forall\ (t, \zeta) \in \mathcal{D}, \tag{6.15}$$

and so the map F in (6.12) is singleton valued.

Example (i). Consider a single-input, single-output system (6.7) of the nonlinear prototype class, with $f_1, f_2\colon \mathbb{R}^2 \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ given by

$$f_1(y, z) = z\sin y, \quad f_2(y, z) = -z|z| + y, \quad g(u) = u^{1/3}. \tag{6.16}$$

As reference signal $r \in \mathcal{R}$, we take the first component $r = \zeta_1$ of the (chaotic) solution of the following Lorenz system of equations.

$$\left.\begin{array}{ll} \dot{\zeta}_1(t) = (1/2)\zeta_2(t) - \zeta_1(t), & \zeta_1(0) = 1, \\ \dot{\zeta}_2(t) = (28/5)\zeta_1(t) - (1/10)\zeta_2(t) - 2\zeta_1(t)\zeta_3(t), & \zeta_2(0) = 0, \\ \dot{\zeta}_3(t) = 2\zeta_1(t)\zeta_2(t) - (8/30)\zeta_3(t), & \zeta_3(0) = 3. \end{array}\right\} \tag{6.17}$$

Recall that the unique global solution of (6.17) is bounded with bounded derivative (see for example [65]). Adopting control parameters $c = 1/4$ and $\varphi\colon t \mapsto 2t$, Figures 6-2(a)-(c) depict the behaviour of the closed-loop system with zero initial state.

Example (ii). To further illustrate the controller strategy (6.9) we simulate a discontinuous feedback strategy in this section for a nonlinear system with $\mathbb{R}^2$-valued input u as follows:

$$\dot{y}(t) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} y(t) + \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} u(t) + y(t)\|y(t)\| + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Big[(\mathcal{B}_{a,b}(y_1))(t) + (\mathcal{B}_{a,b}(y_2))(t)\Big], \tag{6.18}$$

where $\mathcal{B}_{a,b}$ represents a backlash operator (for details, see Section 2.2.4) with parameter $a = \frac{1}{2}$ and initial condition $b = 0$. Set

$$(Ty)(t) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} y + y(t)\|y(t)\| + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Big[(\mathcal{B}_{a,b}(y_1))(t) + (\mathcal{B}_{a,b}(y_2))(t)\Big], \quad t \geq 0,$$

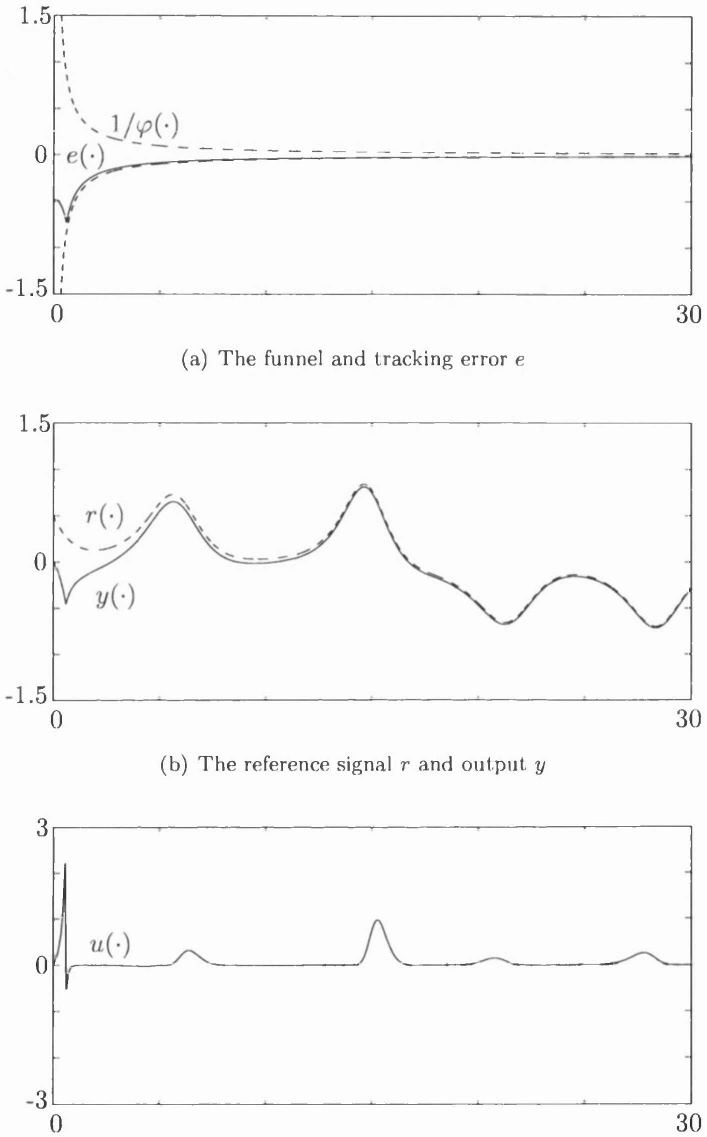
then, writing

$$f(d, w, u) = w + \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} u,$$

system (6.18) can be put in the form (6.1), with $h = 0$. The funnel is specified by the smooth function $t \mapsto \varphi(t) = 2t$ which is such that $\varphi \in \Phi_\lambda$ and a tracking accuracy $\|e(t)\| < 1/(2t)$ for all $t \in [0, \omega)$ is assured. Choosing $\nu\colon k \mapsto k \cos k$ to accommodate for the unknown direction of the control, writing $e(t) = y(t) - r(t)$, setting $\alpha(s) = (1-s^2)^{-1}$ and suppressing the argument t for simplicity, a possible control strategy is

$$\left.\begin{array}{l} u = -k\cos(k)\|e\|^{-1}e, \\ k = \left[1 - \varphi^2\|e\|^2\right]^{-1}. \end{array}\right\} \tag{6.19}$$

For initial data $(y_1(0), y_2(0)) = (1/2, -1/2)$ and reference signals $t \mapsto r_1(t) = \sin 2t$ and $t \mapsto r_2(t) = \cos 2t$, the behaviour of the system (6.18)–(6.19) over the time interval $[0, 10]$ is depicted in Figures 6-3(a)–(c).

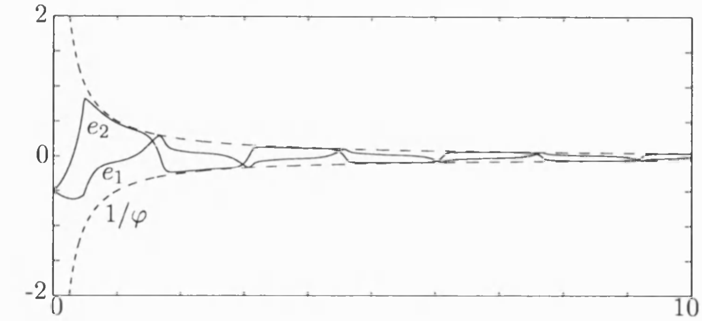


(a) The funnel and tracking error e

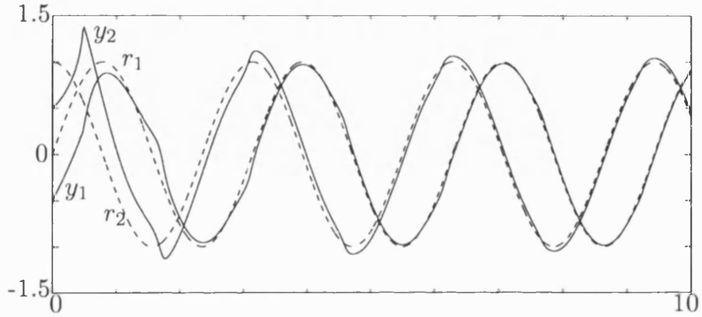
(b) The reference signal r and output y

(c) The control u

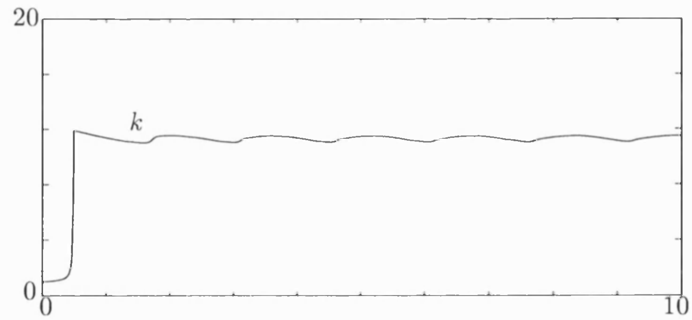
Figure 6-2: Illustration of the continuous control strategy (6.14)–(6.15).



(a) The funnel and tracking error components e_1 and e_2



(b) The reference and output signals. r_1, r_2, y_1 and y_2



(c) The gain function k

Figure 6-3: Behaviour of the system (6.18)–(6.19), tracking $r_1(t) = \sin 2t$ and $r_2(t) = \cos 2t$.

There are, of course, practical issues of synthesis of the control strategy (6.14)–(6.15). Whilst later analysis (see the main result, Theorem 6.6.1) will establish the fact that $\limsup_{t\to\infty} \varphi(t)\|y(t) - r(t)\| < 1$, and so boundedness of the control function u is assured, practical computation of $u(t)$ for large t may encounter numerical ill-conditioning insofar as it involves the product of "large" and "small" quantities (since $\varphi(t) \to \infty$ and $\|y(t) - r(t)\| \to 0$ as $t \to \infty$). These practical issues are not addressed here (the purpose of this work is to highlight those performance characteristics that are attainable in principle): however, we remark that the ill-conditioning associated with the case $\mu = 0$ may be circumvented (at the expense of some degradation in performance) on setting $\lambda > 0$ and replacing unbounded φ by a bounded function $\varphi \in \Phi_\lambda$ with $\liminf_{t\in\mathbb{R}_+} \varphi(t) = 1/\lambda$, in which case, the guaranteed performance is weakened to that of approximate tracking, as quantified by $\limsup_{t\to\infty} \|y(t) - r(t)\| < \lambda$.

6.5.3 Existence of solutions

Here, we will make use of the existence theory developed (with sufficient generality to encompass (6.12)) in Chapter 3.

Theorem 6.5.1 *Let $(f, d, T, h) \in \mathcal{S}$, $\lambda \geq 0$ and $\varphi \in \Phi_\lambda$. Then, for every reference signal $r \in \mathcal{R}$ and all initial data $y^0 \in C([-h, 0], \mathbb{R}^m)$, application of the feedback* (6.10) *to the system* (6.1) *yields the initial-value problem* (6.12) *which has a solution and every solution can be extended to a maximal solution $y\colon [-h, \omega) \to \mathbb{R}^m$, $0 < \omega \leq \infty$. Furthermore, if $y\colon [-h, \omega) \to \mathbb{R}^m$ is a maximal solution and there exist $\sigma > 0$ and a compact $\mathcal{K} \subset \mathcal{D}$ such that $(t, y(t)) \in \mathcal{K}$ for all $t \in [\sigma, \omega)$, then $\omega = \infty$.*

Proof. We identify the initial-value problem (6.12)–(6.13) as a particular case of (3.8) (with $G = F$, $t_0 = 0$ and $\mathcal{D}$ given by (6.11)):

$$\dot{y}(t) \in F(t, y(t), (Ty)(t)), \quad y|_{[-h,0]} = y^0 \in C([-h, 0], \mathbb{R}^m), \quad (0, y^0(0)) \in \mathcal{D}, \qquad (6.20)$$

where

$$F(t, y, w) = \{f(d(t), w, u) \mid u \in -\nu(\alpha(\varphi(t)\|y - r(t)\|))\theta_\mu(y - r(t))\}.$$

An application of Theorem 3.2.1 completes the proof. □

6.6 Main result

We now arrive at the main result, statement (ii) of which asserts that the output of the closed-loop system evolves within the performance funnel and is bounded away from the funnel boundary.

Theorem 6.6.1 *Let $(f, d, T, h) \in \mathcal{S}$, $\lambda \geq 0$ and $\varphi \in \Phi_\lambda$. Then, for every reference signal $r \in \mathcal{R}$ and all initial data $y^0 \in C([-h, 0], \mathbb{R}^m)$, application of the feedback* (6.10) *to the system* (6.1) *yields the closed-loop initial-value problem* (6.12)–(6.13) *which has a solution and each solution can be extended to a maximal solution $y\colon [-h, \omega) \to \mathbb{R}^m$. Every maximal solution $y\colon [-h, \omega) \to \mathbb{R}^m$ has the properties:*

(i) $\omega = \infty$,

(ii) $\sup_{t \in \mathbb{R}_+} \varphi(t)\|y(t) - r(t)\| < 1$,

(iii) *the function $k\colon t \mapsto \alpha(\varphi(t)\|y(t) - r(t)\|)$ is bounded,*

Remark 6.6.2 The conjunction of Assertions (i) and (ii) ensures that both control objectives are attained. Assertion (iii) implies boundedness of the control. In the case where $\varphi(t) \to \infty$ as $t \to \infty$, Assertion (ii) implies asymptotic tracking: $\|y(t) - r(t)\| \to 0$ as $t \to \infty$, that is, asymptotic tracking is assured.

Proof. Let $r \in \mathcal{R}$ and $y^0 \in C([-h, 0], \mathbb{R}^m)$. By Theorem 6.5.1, the closed-loop inital-value problem (6.12)–(6.13) has a solution and every solution can be maximally extended. Let $y\colon [-h, \omega) \to \mathbb{R}^m$ be a maximal solution of (6.12). Defining $e(t) = y(t) - r(t)$ for all $t \in [0, \omega)$, we have

$$\dot{e}(t) + \dot{r}(t) \in F(t, e(t) + r(t), (Ty)(t)) \quad \text{for a.a. } t \in [0, \omega). \tag{6.21}$$

Since $(t, y(t)) \in \mathcal{D}$ for all $t \in [0, \omega)$, it follows that $\varphi(t)\|e(t)\| < 1$ for all $t \in [0, \omega)$. By properties of $\varphi \in \Phi_\lambda$, we may infer boundedness of the function e. Furthermore, since $r \in \mathcal{R}$ is bounded, we may conclude that y is bounded. Invoking Assumptions (S3) and (S4) (in particular, property (iv) of the operator class $\mathcal{T}_h^m$), we deduce the existence of a non-empty, compact set $\mathcal{K} \subset \mathbb{R}^p \times \mathbb{R}^q$ such that $(d(t), (Ty)(t)) \in \mathcal{K}$ for almost all $t \in [0, \omega)$. With this set, we associate the function $\gamma_{\mathcal{K}}$, defined as in (6.2). Writing

$$\Sigma := \{t \in [0, \omega) \mid \|e(t)\| > \mu\} \quad \text{and} \quad k(t) := \alpha(\varphi(t)\|e(t)\|) \quad \forall\, t \in [0, \omega),$$

we have

$$\begin{aligned} t \in \Sigma \quad \Longrightarrow \quad & \langle e(t), f(d(t), (Ty)(t), -\nu(k(t))\|e(t)\|^{-1}e(t))\rangle \\ & \leq -\|e(t)\| \ \min\{\langle u, f(v, w, \nu(k(t))u)\rangle \mid (v,w) \in \mathcal{K}, \ \|u\| = 1\} \\ & = -\|e(t)\|\gamma_{\mathcal{K}}(\nu(k(t))). \end{aligned} \tag{6.22}$$

Noting that

$$t \in \Sigma \quad \Longrightarrow \quad F(t, e(t) + r(t), (Ty)(t)) = \left\{f(d(t), (Ty)(t), -\nu(k(t))\|e(t)\|^{-1}e(t))\right\},$$

we may infer from (6.22) that

$$\langle e(t), v\rangle \leq -\gamma_{\mathcal{K}}(\nu(k(t)))\|e(t)\| \quad \forall\, v \in F(t, e(t) + r(t), (Ty)(t)) \quad \forall\, t \in \Sigma.$$

Therefore, by (6.21) and essential boundedness of $\dot{r}$, there exists $c_0 > 0$ such that

$$\langle e(t), \dot{e}(t)\rangle \leq \left[c_0 - \gamma_{\mathcal{K}}(\nu(k(t)))\right]\|e(t)\| \quad \text{for a.a. } t \in \Sigma. \tag{6.23}$$

By Assumption (S2), either (i) $\limsup_{s\to+\infty} \gamma_{\mathcal{K}}(s) = \infty$, or (ii) $\limsup_{s\to-\infty} \gamma_{\mathcal{K}}(s) = \infty$. Therefore, there exists an unbounded sequence $(s_n) \subset \mathbb{R}$, which is either strictly increasing (in case (i)) or strictly decreasing (in case (ii)), such that the sequence $(\gamma_{\mathcal{K}}(s_n))$ is unbounded and strictly increasing, with $\gamma_{\mathcal{K}}(s_n) > 0$ for all $n \in \mathbb{N}$. By properties (6.8) and continuity of ν, for every $a, b \in \mathbb{R}$ the set $\{\kappa > a \mid \nu(\kappa) = b\}$ is non-empty. Let $k_1 \in \{\kappa > \alpha(\frac{1}{2}) \mid \nu(\kappa) = s_1\}$ be arbitrary and define the strictly-increasing unbounded sequence (k_n) in $(\alpha(\frac{1}{2}), \infty)$ by the recursion $k_{n+1} := \inf\{\kappa > k_n \mid \nu(\kappa) = s_{n+1}\}$, and so $\gamma_{\mathcal{K}}(\nu(k_n)) = \gamma_{\mathcal{K}}(s_n) \to \infty$ as $n \to \infty$.

We proceed to prove boundedness of k. Seeking a contradiction, suppose k is unbounded (in which case, $\mathrm{im}(k) = \mathrm{im}(\alpha) = [\alpha(0), \infty)$). For each $n \in \mathbb{N}$, define

$$\tau_n := \inf\{t \in [0, \omega) \mid k(t) = k_{n+1}\},$$

$$\sigma_n := \sup\{t \in [0, \tau_n] \mid \gamma_{\mathcal{K}}(\nu(k(t))) = \gamma_{\mathcal{K}}(\nu(k_n))\}.$$

We briefly digress to assemble some facts.

Proposition 6.6.3

(a) $\sigma_n < \tau_n \quad \forall\, n \in \mathbb{N}$.

(b) $k(\sigma_n) < k(\tau_n) \quad \forall\, n \in \mathbb{N}$.

(c) $k(t) \geq k_n \quad \forall\, t \in [\sigma_n, \tau_n] \;\; \forall\, n \in \mathbb{N}$.

(d) $\gamma_{\mathcal{K}}(\nu(k(t))) \geq \gamma_{\mathcal{K}}(\nu(k_n)) > 0 \quad \forall\, t \in [\sigma_n, \tau_n] \;\; \forall\, n \in \mathbb{N}$.

(e) $[\sigma_n, \tau_n] \subset \Sigma$ *for all* $n \in \mathbb{N}$.

Proof.
(a) Suppose, for contradiction, that $\sigma_n = \tau_n$ for some $n \in \mathbb{N}$. Then,

$$\gamma_{\mathcal{K}}(s_{n+1}) = \gamma_{\mathcal{K}}(\nu(k_{n+1})) = \gamma_{\mathcal{K}}(\nu(k(\tau_n))) = \gamma_{\mathcal{K}}(\nu(k(\sigma_n))) = \gamma_{\mathcal{K}}(\nu(k_n)) = \gamma_{\mathcal{K}}(s_n),$$

which contradicts strict monotonicity of the sequence $(\gamma_{\mathcal{K}}(s_n))$.

(b) Suppose, for contradiction, that $k(\sigma_n) \geq k(\tau_n) = k_{n+1}$ for some $n \in \mathbb{N}$. Then, since $k(0) = \alpha(0) < \alpha(1/2) < k_{n+1}$, there exists $s \leq \sigma_n < \tau_n$ such that $k(s) = k_{n+1}$, whence the contradiction: $\tau_n = \inf\{t \in [0, \omega) |\; k(t) = k_{n+1}\} \leq s < \tau_n$.

(c) Suppose, for contradiction, that, for some $n \in \mathbb{N}$ and $t \in [\sigma_n, \tau_n]$, $k(t) < k_n$. Then, since $k(\tau_n) = k_{n+1}$, there exists $s \in (\sigma_n, \tau_n]$ such that $k(s) = k_n$. Invoking the definition of σ_n, we arrive at a contradiction: $\sigma_n < s \leq \sigma_n$.

(d) Suppose, for contradiction, that, for some $n \in \mathbb{N}$ and $t \in [\sigma_n, \tau_n]$, $\gamma_{\mathcal{K}}(\nu(k(t))) < \gamma_{\mathcal{K}}(\nu(k_n))$. Since

$$\gamma_{\mathcal{K}}(\nu(k_n)) = \gamma_{\mathcal{K}}(s_n) < \gamma_{\mathcal{K}}(s_{n+1}) = \gamma_{\mathcal{K}}(\nu(k_{n+1})) = \gamma_{\mathcal{K}}(\nu(k(\tau_n))),$$

it follows that, for some $s \in (\sigma_n, \tau_n]$, $\gamma_{\mathcal{K}}(\nu(k(s))) = \gamma_{\mathcal{K}}(\nu(k_n))$, which contradicts the definition of σ_n.

(e) Suppose, for contradiction, that, for some $n \in \mathbb{N}$, there exists $t \in [\sigma_n, \tau_n]$ such that $t \notin \Sigma$, then $\|e(t)\| \leq \mu$. Note that $\alpha(0) < \alpha(\frac{1}{2})$ and, if $\mu > 0$, then $\alpha(\mu\varphi(t)) \leq \alpha(\frac{1}{2})$. Therefore, we arrive at a contradiction.

$$\alpha(\tfrac{1}{2}) < k_n \leq k(t) = \alpha(\varphi(t)\|e(t)\|) \leq \alpha(\tfrac{1}{2}).$$

□

We now return to the proof of Theorem 6.6.1. From assertions (c) and (d) of Proposition 6.6.3, we may infer that

$$\tfrac{1}{2} < \alpha^{-1}(k_n) \leq \alpha^{-1}(k(t)) = \varphi(t)\|e(t)\| < 1 \quad \forall\, t \in [\sigma_n, \tau_n] \;\; \forall\, n \in \mathbb{N}, \tag{6.24}$$

where $\alpha^{-1}\colon [\alpha(0),\infty) \to [0,1)$ is the inverse of the bijection $\alpha\colon [0,1) \to \mathrm{im}(\alpha)$, and

$$-2\varphi^2(t)\|e(t)\|\gamma_{\mathcal{K}}(\nu(k(t)) \le -\varphi(t)\gamma_{\mathcal{K}}(\nu(k(t))) \quad \forall\, t \in [\sigma_n, \tau_n] \;\; \forall\, n \in \mathbb{N}. \tag{6.25}$$

By properties of $\varphi \in \Phi_\lambda$, there exists $c_1 > 0$ such that $\dot\varphi(t) \le c_1[1+\varphi(t)]$ for almost all t which, together with (6.23), yields, for almost all $t \in \Sigma$,

$$\begin{aligned}\frac{d}{dt}\big[\varphi(t)\|e(t)\|\big]^2 &= 2\,\varphi(t)\dot\varphi(t)\,\|e(t)\|^2 + 2\,\varphi^2(t)\,\langle e(t), \dot e(t)\rangle \\ &\le\; 2c_1\,\varphi(t)[1+\varphi(t)]\|e(t)\|^2 + 2\,\varphi^2(t)\|e(t)\|\big(c_0 - \gamma_{\mathcal{K}}(\nu(k(t)))\big).\end{aligned}$$

Invoking (6.24), (6.25) and boundedness of e, we may conclude the existence of $c_2 > 0$ such that

$$\frac{d}{dt}\big[\varphi(t)\|e(t)\|\big]^2 \le \varphi(t)\big[c_2 - \gamma_{\mathcal{K}}(\nu(k(t)))\big] \quad \text{for a.a. } t \in [\sigma_n, \tau_n] \quad \forall n \in \mathbb{N}. \tag{6.26}$$

Fix $n \in \mathbb{N}$ sufficiently large so that $c_2 - \gamma_{\mathcal{K}}(\nu(k_n)) < 0$. Recalling that $\gamma_{\mathcal{K}}(\nu(k(t))) \ge \gamma_{\mathcal{K}}(\nu(k_n))$ for all $t \in [\sigma_n, \tau_n]$, we have

$$\frac{d}{dt}\,[\varphi(t)\|e(t)\|]^2 < 0 \quad \text{for a.a. } t \in [\sigma_n, \tau_n]$$

and so $\varphi(\tau_n)\|e(\tau_n)\| < \varphi(\sigma_n)\|e(\sigma_n)\|$. Therefore,

$$k(\tau_n) = \alpha\big(\varphi(\tau_n)\|e(\tau_n)\|\big) < \alpha\big(\varphi(\sigma_n)\|e(\sigma_n)\|\big) = k(\sigma_n),$$

which contradicts assertion (b) of Proposition 6.6.3. This proves boundedness of k (and so $\nu \circ k\colon t \mapsto \nu(\alpha(\varphi(t)\|y(t) - r(t)\|))$ is also bounded). By boundedness of $t \mapsto k(t) = \alpha(\varphi(t)\|e(t)\|)$, it follows that $\sup_{t\in[0,\omega)} \varphi(t)\|y(t) - r(t)\| < 1$, equivalently, there exists $\varepsilon \in (0,1)$ such that $\varphi(t)\|y(t) - r(t)\| \le 1 - \varepsilon$ for all $t \in [0,\omega)$.

Finally, we show that $\omega = \infty$. By boundedness of y, there exists $c_3 > 0$ such that $\|y(t)\| \le c_3$ for all $t \in [0,\omega)$. Suppose $\omega < \infty$. Then

$$\tilde{\mathcal{K}} := \{(t,v) \in \mathbb{R}_+ \times \mathbb{R}^m \mid \varphi(t)\|v - r(t)\| \le 1 - \varepsilon,\; \|v\| \le c_3,\; t \in [0,\omega]\}$$

is a compact subset of $\mathcal{D}$ with the property $(t, y(t)) \in \tilde{\mathcal{K}}$ for all $t \in [0,\omega)$, which contradicts assertion (iii) of Theorem 3.2.1. Therefore, $\omega = \infty$. This completes the proof. $\square$

6.7 Comparisons with an internal model approach

In [70, Page 210], the *internal model principle* loosely states that every "good" regulator must incorporate a model of the outside world (in the sense that the feedback loop incorporates a suitably reduplicated model of the dynamic structure of the exogenous signals which the closed-loop system is required to track). Revisiting the commentary from Section 1.7.2, in the context of linear systems with linear regulators, "good" means "structurally stable" (see [70, 67]) and "good" amounts to a "signal detection" property in a more general context of smooth nonlinear systems (see, [62]). The absence of an internal model in the feedback structure proposed in this chapter leads us to conclude that the closed-loop system of Section 6.5.1 lacks robustness in some sense.

The perceived lack of robustness in the control strategy may stem from the potential singularity introduced via the injection α in the closed loop or from the unbounded nature of the funnel function φ. It is not unreasonable to expect that the adoption of a bounded function φ (with attendant reduction in performance from asymptotic to approximate tracking) might induce some robustness in the closed loop. However, in the absence of a rigorous robustness analysis, the results in this chapter are mainly of a theoretical nature, serving to illustrate those performance characteristics that are attainable, in principle, under weak assumptions on the plant data.

Chapter 7

Asymptotic tracking for systems with input hysteresis

In Chapter 6, a large class of multi-input, multi-output nonlinear systems was investigated with two control objectives, namely asymptotic tracking and prescribed transient behaviour. The generality afforded by Assumption (S2) of the system class $\mathcal{S}$, allowing for input nonlinearities that could affect the polarity of the input signal in a manner unpredictable by a controller, was discussed in Section 6.4.1. In the current chapter, a class of single-input, single-output systems is considered which allows for hysteretic effects in the input channel.

7.1 Introduction

In [22], single-input, single-output nonlinear systems of the form

$$\dot{y}(t) = f(p(t), (Ty)(t)) + \beta\,(\Phi u)(t), \quad y|_{[-h,0]} = y^0 \in C([-h,0],\mathbb{R}) \tag{7.1}$$

are examined, where $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is assumed to be locally Lipschitz in its second argument, $p \in L^\infty(\mathbb{R}_+, \mathbb{R})$ is a perturbation, T is a causal operator, $h \geq 0$ quantifies the memory of the system, $\beta \in \mathbb{R}$ is a non-zero real parameter and Φ is a hysteresis operator. The control objectives are approximate tracking and prescribed transient behaviour. In the present chapter, we combine the ideas of Chapter 6 and [22] by applying a variant of the controller studied in Chapter 6 to a system of the form (7.1). The aim will be to ensure *asymptotic* tracking with prescribed transient performance.

Hysteresis in systems and control has recently received considerable attention in a variety of applications, see for example the control of hysteresis in smart actuators in [66] and tracking control using piezoceramic actuators in [18], for example. Systems subject to input hysteresis are considered in [40] and [42] as well as [22], the inspiration for this chapter.

We consider single-input, single-output, nonlinear systems, modelled by functional differential equations of the form (7.1), where β and h are as before, $p \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$, $T \in \mathcal{T}_h^1$ and $f \in C(\mathbb{R}^m \times \mathbb{R}^q, \mathbb{R})$ is locally Lipschitz in its second argument. The class of reference signals to be tracked is given by $\mathcal{R} := W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$.

This chapter is structured as follows. In Section 7.2, the control objectives are detailed and a full description of the system class (and several subclasses) is provided in Section 7.3. The feedback control strategy and resulting closed-loop system are given in Section 7.4 and an example can be found in Section 7.4.2. Finally, after the development of a suitable existence theory in Section 7.4.3, the main result of the chapter can be found in Section 7.5.

7.2 Control objectives and the performance funnel

The two control objectives considered here match the aims of Chapter 6 (in the single-input, single-output case):

(i) tracking of any reference signal $r \in \mathcal{R}$ by the output y. For arbitrary $\lambda \geq 0$, we seek an output feedback strategy which ensures that, for every $r \in \mathcal{R}$, the closed-loop system has bounded solution and the tracking error $e = y - r$ is such that $\limsup_{t\to\infty} \|e(t)\| < \lambda$ if $\lambda > 0$ or $\lim_{t\to\infty} \|e(t)\| = 0$ if $\lambda = 0$;

(ii) prescribed transient behaviour of the tracking error.

Both objectives are again captured in the concept of a performance funnel

$$\mathcal{F}_\varphi := \{(t,e) \in \mathbb{R}_+ \times \mathbb{R} \,|\, \varphi(t)\,|e| < 1\}$$

associated with a function φ belonging to Φ_λ, introduced in Section 1.3, viz.

$$\Phi_\lambda = \Big\{\varphi \in AC_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}_+) \;\Big|\; \varphi(0) = 0,\ \varphi(s) > 0 \ \ \forall s > 0,\ \liminf_{s\to\infty} \varphi(s) \geq 1/\lambda,$$
$$\exists\, c > 0 : \ \ \dot\varphi(s) \leq c[1 + \varphi(s)] \text{ for a.a. } s > 0\Big\},$$

with the convention that, if $\lambda = 0$, then $1/\lambda := \infty$, in which case φ is unbounded and $\varphi(t) \to \infty$ as $t \to \infty$. The case $\lambda = 0$ is of particular interest since, in this situation, evolution within the funnel would imply that $e(t) \to 0$ as $t \to \infty$ and so an asymptotic tracking objective is attained.

The aim, as in previous chapters, will be to devise a feedback structure which ensures that, for every system of the underlying class and every reference signal $r \in \mathcal{R}$, the graph of the tracking error $e = y - r$ is properly contained in $\mathcal{F}_\varphi$, in the sense that $\sup_{t\in\mathbb{R}_+} \varphi(t)|e(t)| < 1$ then the tracking objective (i) is attained and (ii) prescribed transient behaviour is dictated by the choice of φ.

7.3 Class of systems

7.3.1 Input hysteresis class

Recall the definition of a causal, rate-independent hysteresis operator (see Definition 2.2.3). We make precise the class of hysteresis operators that will be considered in the input channel.

Definition 7.3.1 (Class $\mathcal{O}$ of hysteresis operators)
A causal and rate-independent operator $\Phi\colon C(\mathbb{R}_+,\mathbb{R}) \to C(\mathbb{R}_+,\mathbb{R})$ is said to be of class $\mathcal{O}$ if the following hold.

(i) *There exists $c_0 > 0$ such that, for all $t \geq 0$ and all $w \in C([0,t],\mathbb{R})$, there exist $\tau > t$ and $\delta > 0$ such that*

$$\sup_{t\in[0,\tau]} |(\Phi u_1)(t) - (\Phi u_2)(t)| \leq c_0 \sup_{t\in[0,\tau]} |u_1(t) - u_2(t)| \quad \forall\, u_1, u_2 \in \mathcal{C}(w;0,t,\tau,\delta)\,.$$

(ii) *For all $\rho > 0$ and all $u \in C([0,\rho),\mathbb{R})$, there exists $c_1 > 0$ such that*

$$\sup_{s\in[0,t]} |(\Phi u)(s)| \leq c_1(1 + \sup_{s\in[0,t]} |u(s)|) \quad \forall t \in [0,\rho).$$

(iii) *There exist $c_2 > 0$ and $c_3 > 0$ such that, for all $u \in C(\mathbb{R}_+,\mathbb{R})$ and all $t \in \mathbb{R}_+$,*

$$|u(t)| \geq c_2 \quad \Longrightarrow \quad c_3 u^2(t) \leq u(t)\,(\Phi u)(t).$$

Remark 7.3.2

(i) To interpret (i) and (ii) correctly, recall the localization procedure described in Remark 2.1.2, see Section 2.1.

(ii) Assumption (iii) is a weak sector-bounded condition that will be utilized in the analysis later in this chapter. We remark that the hysteretic effects described in

Chapter 2 such as the backlash, Preisach and Prandtl operators satisfy assumption (iii) and are in the class $\mathcal{O}$, see Section 7.3.3.

7.3.2 System class

Definition 7.3.3 *(Hysteretic system class $\mathcal{H}$)*
The class of systems $\mathcal{H}$ is comprised of single-input, single-output nonlinear systems $(f, p, T, \beta, \Phi, h)$ of the form (7.1) satisfying the following assumptions:

(H1) $f\colon \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}$ *is continuous and* $f(z, \cdot)$ *is locally Lipschitz for every* $z \in \mathbb{R}$*;*

(H2) $p \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$*;*

(H3) $T \in \mathcal{T}_h^1$*;*

(H4) $\beta \in \mathbb{R}$ *is non-zero;*

(H5) $\Phi \in \mathcal{O}$*.*

Remark 7.3.4 With reference to Figure 7-1, a system (7.1) of class $\mathcal{H}$ can be thought of in terms of interconnected subsystems Λ_1 and Λ_2, with Λ_1 driven by a perturbation p, the input signal $v = \Phi u$ and the output w from the system Λ_2. System Λ_2 is formulated as a causal operator mapping the system output y to w.

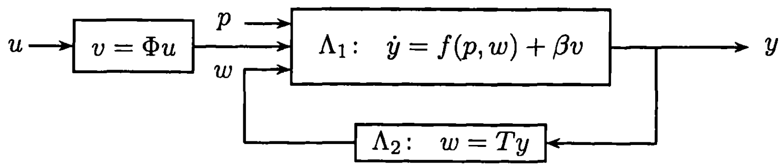


Figure 7-1: System of class $\mathcal{H}$.

The setup bears a resemblance to the structure considered in Chapter 6 (see Figure 6-1), however, the key difference in this work (and in the paper [22]) is the cascade formed by the hysteresis operator (of class $\mathcal{O}$) acting on the input.

7.3.3 Subclasses of $\mathcal{H}$

We first examine a class of linear systems that form one of the more basic subclasses of $\mathcal{H}$.

Linear prototype

Consider the particular subclass of $\mathcal{L}$ (see Section 1.4.4) comprised of finite-dimensional, minimum-phase, single-input ($u(t) \in \mathbb{R}$), single-output ($y(t) \in \mathbb{R}$) linear systems (A, b, c) of relative degree one and introduce an input nonlinearity $\Phi \in \mathcal{O}$. Consider the following:

$$\dot{x}(t) = Ax(t) + b\,(\Phi u)(t), \quad x(0) = x^0. \tag{7.2}$$

As in Section 1.4.5 there exists a similarity transformation which takes system (7.2) into the form

$$\left.\begin{aligned} \dot{y}(t) &= A_1 y(t) + A_2 z(t) + cb\,(\Phi u)(t), \qquad & y(0) &= y^0, \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t), & z(0) &= z^0, \end{aligned}\right\} \tag{7.3}$$

where, by the minimum-phase property, A_4 is a Hurwitz matrix. Defining the function p and operator T, as before, by

$$p(t) := A_2\big(\exp(A_4 t)\big)z^0 \quad \text{and} \quad (Ty)(t) := A_1 y(t) + A_2 \int_0^t \big(\exp A_4(t-s)\big) A_3 y(s)ds\,,$$

we see that the original system (7.2) can be recast in the form of a functional differential equation

$$\dot{y}(t) = p(t) + (Ty)(t) + cb\,(\Phi u)(t), \quad y(0) = y^0 \in \mathbb{R}\,,$$

which is of the form (7.1) with $h = 0$, $\beta = cb$ and $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, w) \mapsto x + w$. Clearly, Assumptions (H1), (H4) and (H5) hold. Recalling that A_4 is Hurwitz, we see that (H2) and (H3) (with parameter $h = 0$) are valid. Consequently the system is of class $\mathcal{H}$.

Nonlinear systems

We also highlight a particular subclass of $\mathcal{H}$ that will be utilized in an example. Consider a class of single-input ($u(t) \in \mathbb{R}$), single-output ($y(t) \in \mathbb{R}$), nonlinear systems of the form

$$\left.\begin{aligned} \dot{y}(t) &= f_1(y(t), z(t)) + \beta\,(\Phi u)(t), \qquad & y(0) &= y^0 \in \mathbb{R}, \\ \dot{z}(t) &= f_2(y(t), z(t)), & z(0) &= z^0 \in \mathbb{R}^p, \end{aligned}\right\} \tag{7.4}$$

with f_1 and f_2 locally Lipschitz, $\Phi \in \mathcal{O}$ and $\beta \neq 0$. Temporarily regarding y as an independent input to the second subsystem in (7.4) and following the procedure from Section 2.2.2, denote the unique solution of the initial-value problem $\dot{z} = f_2(y, z)$, $z(0) = z^0$, by $z(\cdot, z^0, y)$. Assume that the second subsystem in (7.4) is input-to-state stable (ISS) (see [60] or the commentary in Section 2.2.2), then, for each $z^0 \in \mathbb{R}^p$, we may define an operator $C(\mathbb{R}_+, \mathbb{R}) \to C(\mathbb{R}_+, \mathbb{R} \times \mathbb{R}^p)$ by

$$(Ty)(t) := (y(t), z(t, z^0, y)) \quad \forall\, t \in \mathbb{R}_+.$$

This operator T is of class $\mathcal{T}_0^1$ (Assumption (H3) holds with $h = 0$). System (7.4) may be expressed as functional differential equation

$$\dot{y}(t) = f_1((Ty)(t)) + \beta\,(\Phi u)(t), \quad y(0) = y^0,$$

which is of the form (7.1) with $h = 0$ and $f\colon (x, w) \mapsto f_1(w)$. Evidently, Assumptions (H1), (H4) and (H5) hold and Assumption (H2) is vacuous, therefore (7.4) is of class $\mathcal{H}$.

Delays and hysteresis

Recall, from Chapter 2, that nonlinear delay elements are incorporated in the operator class $\mathcal{T}_h^1$, whilst the class $\mathcal{T}_0^1$ encompasses a wide range of hysteresis operators, including many physically motivated effects: such as relay hysteresis, backlash hysteresis, elastic-plastic hysteresis and Preisach operators (see Section 2.2.4).

In addition, the class $\mathcal{O}$ of input nonlinearities also incorporates a wide range of interesting nonlinear effects. Clearly, Property (iii) of class $\mathcal{O}$ imposes a significant restriction when compared with the generality of $\mathcal{T}_h^m$, though hysteresis phenomena such as relay and backlash operators meet this additional criterion. In particular, the Preisach operator also satisfies Property (iii); a proof of this fact can be found in [22, Appendix 1].

7.4 Output feedback

Let $\nu\colon \mathbb{R} \to \mathbb{R}$ be any locally Lipschitz function with the properties

$$\limsup_{k\to\infty} \nu(k) = +\infty \quad \text{and} \quad \liminf_{k\to\infty} \nu(k) = -\infty\,. \tag{7.5}$$

Let $\alpha\colon [0,1) \to \mathbb{R}_+$ be a locally Lipschitz, unbounded injection; the example $\alpha\colon s \mapsto 1/(1-s)$ provided in Chapter 6 suffices. For $r \in \mathcal{R}$, $\lambda > 0$, $\varphi \in \Phi_\lambda$, consider the control strategy

$$\left.\begin{aligned} u(t) &= \nu(k(t))\varphi(t)(y(t) - r(t)), \\ k(t) &= \alpha(\varphi(t)|y(t) - r(t)|). \end{aligned}\right\} \tag{7.6}$$

Remark 7.4.1

(i) Note that the additional restriction imposed on α, namely that the function be locally Lipschitz, is necessary to ensure the existence of a solution to the closed-loop system discussed in the next section. This assumption was not required on the corresponding function in the feedback developed in Section 6.5.

(ii) We will aim to prove that the feedback (7.6), applied to the cascade in Figure 7-1, given by (7.1), achieves the specified control objectives. The output feedback controller utilized in [22] is given by

$$\begin{aligned} u(t) &= \nu(k(t))(y(t) - r(t)), \\ k(t) &= \alpha(\varphi(t)|y(t) - r(t)|). \end{aligned}$$

The key difference in the control (7.6), when compared to the controller in [22] is the explicit presence of φ. The potential difficulty that may be faced in the case of unbounded φ is addressed in Section 7.4.2.

7.4.1 Closed-loop system

Let $\lambda \geq 0$, $\varphi \in \Phi_\lambda$, $r \in \mathcal{R}$ and let $\mathcal{D} \subset \mathbb{R}_+ \times \mathbb{R}$ denote the set

$$\{(t, y) \in \mathbb{R}_+ \times \mathbb{R} \mid \ \varphi(t)|y - r(t)| < 1\}. \tag{7.7}$$

Let $(f, p, T, \beta, \Phi, h) \in \mathcal{H}$. The conjunction of (7.6) with (7.1) yields the following closed-loop initial-value problem

$$\left.\begin{aligned} \dot{y}(t) &= f(p(t), (Ty)(t)) + \beta\,(\Phi u)(t), \qquad y|_{[-h,0]} = y^0 \in C([-h, 0], \mathbb{R}), \\ u(t) &= \nu(k(t))\varphi(t)(y(t) - r(t)), \\ k(t) &= \alpha(\varphi(t)|y(t) - r(t)|). \end{aligned}\right\} \tag{7.8}$$

By a solution of (7.8), we mean a continuous function $y\colon I \to \mathbb{R}$ on an interval of the form $[-h, \rho]$ with $0 < \rho < \infty$ or of the form $[-h, \omega)$, with $0 < \omega \leq \infty$, such that (a) $y|_{[-h,0]} = y^0$ and (b) y_J, $J := I \backslash [-h, 0)$, is a locally absolutely continuous function, with graph in $\mathcal{D}$ and satisfying the differential equation in (7.8) almost everywhere on J.

As in previous Chapters, we shall demonstrate that the control objectives are achieved by establishing that: (i) the initial-value problem (7.8) has a solution; (ii) every solution can be extended to a maximal solution; (iii) every maximal solution is global. Facts (i) and (ii) are a consequence of the existence theory developed in Section 7.4.3 below, whilst fact (iii) is the focus of the main result, Theorem 7.5.1.

7.4.2 Commentary on the asymptotic tracking problem

Consider the case $\lambda = 0$, in which an exact asymptotic tracking objective is sought (that is, for every $r \in \mathcal{R}$, $y(t) - r(t) \to 0$ as $t \to \infty$). In Chapter 6 it was shown that the control strategy developed was capable of ensuring an asymptotic tracking objective via continuous feedback, provided an appropriate choice of α was utilized (see Section 6.5.2). For comparison, we illustrate, in this section, the continuous feedback control strategy (7.6) for a single-input, single-output system (7.4) of the nonlinear prototype class, with $\beta \neq 0$ and $f_1, f_2 \colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f_1(y,z) = z \sin y, \quad f_2(y,z) = -z|z| + y.$$

Let the input nonlinearity Φ be given by a backlash hysteresis operator $\mathcal{B}_{a,b}$ with parameters $a > 0$ and $b \in [-a, a]$ (for a full description, see Section 2.2.4).

Choosing $\varphi \colon t \mapsto 2t$, $\nu \colon k \mapsto k\cos(ck)$ and $\alpha \colon s \mapsto 1/(1-s^2)$, the control strategy takes the form:

$$u(t) = \psi(t, y(t) - r(t)), \tag{7.9}$$

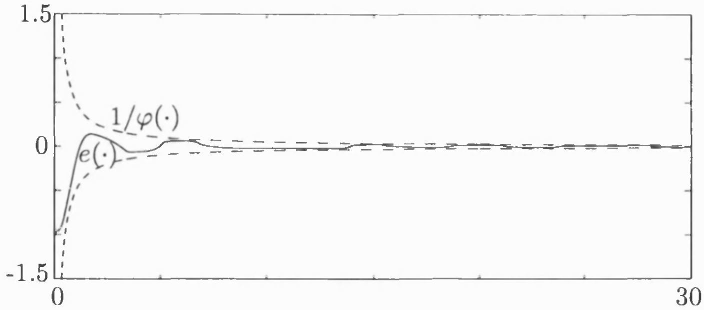
with $\psi \in C(\mathcal{D}, \mathbb{R})$ given by

$$\psi(t,\zeta) := \cos\left(\frac{c}{1-\varphi^2(t)|\zeta|^2}\right)\left(\frac{\varphi(t)\zeta}{1-\varphi^2(t)|\zeta|^2}\right) \quad \forall\ (t,\xi) \in \mathcal{D}. \tag{7.10}$$

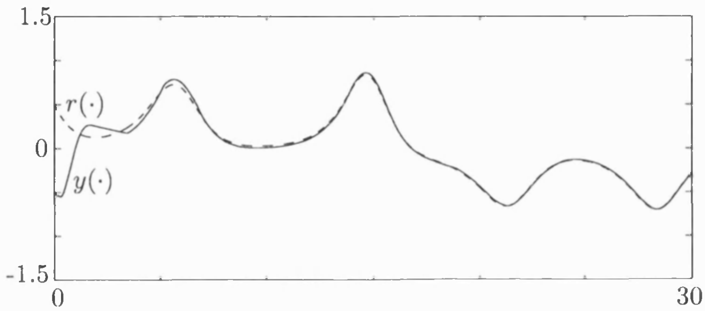
As reference signal $r \in \mathcal{R}$, we take the first component $r = \zeta_1$ of the (chaotic) solution of the Lorenz system of equations given by (6.17).

With initial state $(y(0), z(0)) = (-1/2, 0)$ and parameters $a = 1$, $b = 0$, $c = 1/4$ and $\beta = 1$, Figures 7-2(a)–(c) depict the behaviour of the closed-loop system.

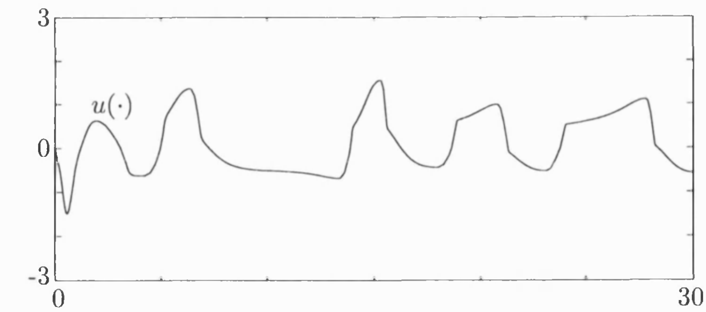
Synthesis of the control strategy (7.9)–(7.10) could face the same practical issues raised in Section 6.5.2. Though it will be established in later analysis (see Theorem 7.5.1) that $\limsup_{t\to\infty} \varphi(t)|y(t) - r(t)| < 1$ and that k (and hence u) is bounded, numerical ill-conditioning may occur in the computation of $u(t)$ for large t. We remark, once more, that this potential difficulty can be circumvented by setting $\lambda > 0$ and replacing φ by a bounded function $\varphi \in \Phi_\lambda$, in which case, a weaker, approximate tracking objective is ensured.



(a) The funnel and tracking error e



(b) The reference r and output y



(c) The control u

Figure 7-2: Control (7.9)-(7.10) applied to the nonlinear system (7.4).

7.4.3 Existence theory

Let $(f,p,T,\beta,\Phi,h) \in \mathcal{H}$, let $\lambda > 0$, $\varphi \in \Phi_\lambda$, $r \in \mathcal{R}$ and let $\mathcal{D} \subset \mathbb{R}_+ \times \mathbb{R}$ be defined as in (7.7). Consider the following family of initial-value problems, parameterized by $t_0 \in \mathbb{R}_+$,

$$\left.\begin{aligned} &\dot{y}(t) = f(p(t),(Ty)(t)) + \beta\,(\Phi u)(t), \quad y|_{[-h,t_0]} = y^0 \in C([-h,t_0],\mathbb{R}),\\ &u(t) = \nu(k(t))\varphi(t)(y(t)-r(t)),\\ &k(t) = \alpha(\varphi(t)|y(t)-r(t)|). \end{aligned}\right\} \tag{7.11}$$

By a solution of (7.11) we mean the generalization of the earlier concept of a solution: a continuous function $y\colon I \to \mathbb{R}$ on an interval of the form $[-h,\rho]$ with $t_0 < \rho < \infty$ or of the form $[-h,\omega)$, with $t_0 < \omega \le \infty$, such that (a) $y|_{[-h,t_0]} = y^0$ and (b) y_J, $J := I \setminus [-h,t_0)$, is a locally absolutely continuous function, with graph in $\mathcal{D}$ and satisfying the differential equation in (7.11) almost everywhere on J. We will prove the following.

Theorem 7.4.2 *For every $t_0 \in \mathbb{R}_+$ and every $y^0 \in C([-h,t_0],\mathbb{R})$ with $(t,y^0(t)) \in \mathcal{D}$ for all $t \in [0,t_0]$, the initial-value problem (7.11) has a unique maximal solution $y \in C[-h,\omega)$. Furthermore, if $\omega < \infty$, then $\limsup_{t\to\omega}\varphi(t)|y(t)-r(t)| = 1$ (or equivalently, $\limsup_{t\to\omega} k(t) = \infty$).*

The proof of Theorem 7.4.2 contains only minor modifications to the proof of [22, Theorem 7.1] and so is relegated to the appendix, see Section B.2.

Remark 7.4.3 Note that only properties (i) and (ii) of the class of input nonlinearities $\mathcal{O}$ are required in the proof of Theorem 7.4.2. Additionally imposing condition (iii) of $\mathcal{O}$ in the main result, below, will guarantee that, for each maximal solution $y \in C([-h,\omega),\mathbb{R})$ of (7.8), $\omega = \infty$.

7.5 Main result

Theorem 7.5.1 *Let $(f,p,T,\beta,\Phi,h) \in \mathcal{H}$, $\lambda > 0$ and $\varphi \in \Phi_\lambda$. Then for every $r \in \mathcal{R}$ and all initial data $y^0 \in C([-h,0],\mathbb{R})$, application of the feedback (7.6) to the system (7.1) yields the closed-loop initial-value problem (7.8) which has a unique maximal solution $y\colon [-h,\omega) \to \mathbb{R}$. Each maximal solution $y\colon [-h,\omega) \to \mathbb{R}$ is such that:*

(i) $\omega = \infty$,

(ii) there exists $\varepsilon \in (0,1)$ such that, for all $t \in \mathbb{R}_+$, $\varphi(t)|y(t)-r(t)| \le 1-\varepsilon$,

(iii) the continuous functions $u, \Phi u\colon \mathbb{R}_+ \to \mathbb{R}$ and $k\colon \mathbb{R}_+ \to \mathbb{R}_+$ are bounded.

Remark 7.5.2 Observe that, in the case when $\varphi(t) \to \infty$ as $t \to \infty$, assertion (ii) of Theorem 7.5.1 implies that $|y(t) - r(t)| \to 0$ as $t \to \infty$, that is, asymptotic tracking is assured.

Proof. Let $r \in \mathcal{R}$ and $y^0 \in C([-h, 0], \mathbb{R})$ be arbitrary. Applying Theorem 7.4.2 for the special case in which $t_0 = 0$ yields the existence of a unique maximal solution $y\colon [-h, \omega) \to \mathbb{R}$ of (7.8), with $0 < \omega \leq \infty$. Observe that, since $(t, y(t)) \in \mathcal{D}$ for all $t \in [0, \omega)$ and r is bounded, it follows that y is bounded. By property (iv) of the operator class $\mathcal{T}_h^1$, the function Ty is bounded. Writing

$$e(t) := y(t) - r(t), \quad k(t) = \alpha(\varphi(t)|e(t)|), \quad u(t) = \nu(k(t))\varphi(t)e(t) \quad \forall t \in [0, \omega),$$

we have

$$\dot{e}(t) = f(p(t), (Ty)(t)) + \beta\,(\Phi u)(t) - \dot{r}(t), \quad \text{for a.a. } t \in [0, \omega).$$

By boundedness of e, continuity of f, boundedness of Ty and since p and $\dot{r}$ are essentially bounded, there exists $c_0 > 0$ such that

$$e(t)\dot{e}(t) \leq c_0|e(t)| + \beta e(t)\,(\Phi u)(t) \quad \text{for a.a. } t \in [0, \omega). \tag{7.12}$$

Hence, by properties of φ, there exist $c_1, c_2 > 0$ such that

$$\begin{aligned}\frac{d}{dt}(\varphi(t)e(t))^2 &= 2\varphi(t)\dot{\varphi}(t)e^2(t) + 2\varphi^2(t)e(t)\dot{e}(t)\\ &\leq 2c_1\varphi(t)(1 + \varphi(t))e^2(t) + 2\varphi^2(t)e(t)\dot{e}(t)\\ &\leq \varphi(t)(c_2 + 2\beta\varphi(t)e(t)\,(\Phi u)(t)).\end{aligned} \tag{7.13}$$

We proceed by showing that k is bounded. Seeking a contradiction, suppose that k is unbounded. By properties (7.5) of ν, there exists a strictly-increasing unbounded sequence (k_n) in $(\alpha(1/2), \infty)$ such that $(\beta\nu(k_n))$ is a strictly-decreasing unbounded sequence in $(-\infty, 0)$. For each $n \in \mathbb{N}$, define

$$\tau_n := \inf\{t \in [0, \omega) \mid k(t) = k_{n+1}\}, \quad \sigma_n := \sup\{t \in [0, \tau_n] \mid \nu(k(t)) = \nu(k_n)\} < \tau_n,$$

wherein the latter inequality holds since $|\nu(k(\tau_n))| = |\nu(k_{n+1})| > |\nu(k_n)|$. We collect the following facts for later use.

$$\left.\begin{array}{l} k_n(t) \le k(t) \quad \text{and} \quad |\nu(k_n)| \le |\nu(k(t))| \\ \varphi(t)|e(t)| = \alpha^{-1}(k(t)) \ge \alpha^{-1}(k_n) > 1/2 \end{array}\right\} \quad \forall t \in [\sigma_n, \tau_n] \quad \forall n \in \mathbb{N}, \tag{7.14}$$

where α^{-1} denotes the inverse of the bijection $\alpha\colon [0,1) \to [\alpha(0), \infty)$. Property (iii) of the class of nonlinearities $\mathcal{O}$ states that there exist $\Delta, \delta > 0$ such that, for all $u \in C(\mathbb{R}_+, \mathbb{R})$ and all $t \in \mathbb{R}_+$,

$$|u(t)| \ge \Delta \quad \Longrightarrow \quad \delta u^2(t) \le u(t)\,(\Phi u)(t).$$

There exists $N \in \mathbb{N}$ sufficiently large so that $(1/2)|\nu(k_N)| \ge \Delta$. By (7.14), it follows that

$$|u(t)| = |\nu(k(t))\varphi(t)e(t)| \ge (1/2)|\nu(k_N)| \ge \Delta \quad \forall t \in [\sigma_n, \tau_n] \quad \forall n > N$$

and so, for all $t \in [\sigma_n, \tau_n]$ and all $n > N$

$$\nu(k(t))\varphi(t)e(t)\,(\Phi u)(t) = u(t)\,(\Phi u)(t) \ge \delta u^2(t) = \delta(\nu(k(t)))^2\varphi^2(t)e^2(t). \tag{7.15}$$

Since $\beta\nu(k(t)) \le \beta\nu(k_n) < 0$ for all $t \in [\sigma_n, \tau_n]$ and all $n \in \mathbb{N}$, multiplying each side of (7.15) by $\beta/\nu(k(t))$, we may conclude, from (7.14) and (7.15), that

$$\beta\varphi(t)e(t)\,(\Phi u)(t) \le \beta\varphi^2(t)e^2(t)\delta\nu(k(t)) \le \beta\varphi^2(t)e^2(t)\delta\nu(k_n) \quad \forall t \in [\sigma_n, \tau_n] \quad \forall n > N$$

which, in conjunction with (7.13), yields

$$\frac{d}{dt}(\varphi(t)e(t))^2 \le \varphi(t)(c_2 + 2\beta\varphi^2(t)e^2(t)\delta\nu(k_n)) \tag{7.16}$$

$$\le \varphi(t)(c_2 + 1/2\beta\delta\nu(k_n)) \quad \forall t \in [\sigma_n, \tau_n] \quad \forall n > N. \tag{7.17}$$

Fix $n^* > N$ sufficiently large so that

$$c_2 + 1/2\beta\delta\nu(k_n) < 0,$$

then by (7.16), we have

$$\varphi(\tau_{n^*})|e(\tau_{n^*})| < \varphi(\sigma_{n^*})|e(\sigma_{n^*})|,$$

whence the contradiction

$$k(\tau_{n^*}) = \alpha\big(\varphi(\tau_{n^*})|e(\tau_{n^*})|\big) < \alpha\big(\varphi(\sigma_{n^*})|e(\sigma_{n^*})|\big) = k(\sigma_{n^*}),$$

proving boundedness of k.

By boundedness of $t \mapsto k(t) = \alpha(\varphi(t)|e(t)|)$, it follows that, for some $\varepsilon \in (0,1)$, $\varphi(t)|e(t)| < 1 - \varepsilon$ for all $t \in [0, \omega)$. By Theorem 7.4.2, it follows that $\omega = \infty$. Finally, boundedness of k and the product φe ensures boundedness of the control u. By property (ii) of the hysteresis class $\mathcal{O}$, it follows that Φu is also bounded. This completes the proof. $\square$

Chapter 8

Conclusions

8.1 Concluding remarks

In this thesis, the problem of developing universal controllers, capable of influencing both the transient and asymptotic behaviour of solutions of various classes of functional differential equations, was addressed. Many of the systems considered involved a class of nonlinear, causal operators that were shown to allow a diverse range of phenomena to be incorporated.

The controllers designed in this thesis involved continuous output feedback in all cases with the exception of the strategy developed in Chapter 6, in which a potentially discontinuous controller was implemented, interpreted within the framework of a differential inclusion.

The presence of the nonlinear operator and the potentially discontinuous control strategy adopted in Chapter 6 necessitated the development of suitable existence theorems for functional differential equations and inclusions with sufficient generality to encompass each of the systems considered.

Chapters 4–7 in this thesis considered four different system classes. The first examination involved multi-input, multi-output, nonlinearly-perturbed, linear systems of known relative degree and this was followed by an investigation of a larger class of multi-input, multi-output, nonlinear systems modelled by controlled functional differential equations, also having known relative degree. In Chapter 6, multi-input, multi-output systems of relative degree 1 were considered, but the restriction on the relative degree was counter-balanced by enhanced control aims and potentially nonlinear effects in the input channel. Finally, in Chapter 7, single-input, single-output, nonlinear systems, described by functional differential equations, were examined, with hysteretic effects permissible in the input channel.

For each of the four classes considered, the main requirements were:

(i) the resulting initial-value problem must have a global solution;

(ii) the state variables, gain function and control should remain bounded;

(iii) the control objectives must be attained.

The first control objective in each of the main chapters was tracking of a reference signal by the output of the system considered. In Chapters 4 and 5, the objective was *approximate* tracking, whilst in Chapters 6 and 7, the objective also incorporated *asymptotic* tracking. The second control objective, namely prescribed transient behaviour of the tracking error signal, was shared by all of the four main chapters.

The control objectives were captured in the concept of a prescribed performance funnel. The feedback structures developed in this thesis essentially exploited an intrinsic high-gain property of the systems examined by ensuring that, as the error approached the funnel boundary, the gain function attained values sufficiently large so as to preclude boundary contact.

Finally, the results of each chapter were illustrated with simulations of simple examples.

8.2 Further work

- The problem of *asymptotic* tracking for systems of known relative degree $\rho > 1$ was not tackled in this thesis and could play a rôle in future study involving funnel control.
- The nonlinear class of systems considered in Chapter 5 were affine in the control. It may be of interest to extend the investigations in this thesis to systems with nonlinearities (or even hysteretic effects in the single-input, single-output case) in the input channel.
- The approach of [24] involves an internal model described by a linear system of equations. It may be possible to consider an expanded class of reference signals, in the context of a funnel control problem, by adopting a nonlinear internal model, as in [7] and [53], for example.

Appendix A

Background results

In this section of the appendix, several key definitions as well as the statements of important results referred to in this thesis are provided.

A.1 Basic definitions

Let I be a closed, bounded interval.

Definition A.1.1 *Let $(X, \|\cdot\|_X)$ be a metric space. A set $A \subset X$ is relatively compact if the closure of A is compact.*

Definition A.1.2 *A family $\mathcal{F} \subset C(I, \mathbb{R}^n)$ is said to be uniformly bounded if there exists $c > 0$ such that*

$$\|x(t)\| \leq c \quad \forall t \in I \quad \forall x \in \mathcal{F}.$$

Definition A.1.3 *A family $\mathcal{F} \subset C(I, \mathbb{R}^n)$ is said to be equicontinuous if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $s, t \in I$,*

$$|s - t| < \delta \Longrightarrow \|x(s) - x(t)\| < \varepsilon \quad \forall x \in \mathcal{F}.$$

A.2 Background results

Theorem A.2.1 (Arzelà Ascoli) *$\mathcal{F} \subset C(I, \mathbb{R}^n)$ is relatively compact if, and only if, $\mathcal{F}$ is bounded and equicontinuous.*

Theorem A.2.2 (Alaoglu's Theorem)
If X is a normed vector space, the closed unit ball $\{f \in X^* \mid \|f\| \leq 1\}$ in X^* is compact under the weak topology.*

For a proof, see [17, Theorem 5.18].

Lemma A.2.3 (Gronwall's Lemma)

Let $\rho > 0$, $\phi \in L^1([t,t+\rho],\mathbb{R})$ and $\psi \in AC([t,t+\rho],\mathbb{R})$ with the property that $\phi(t) \geq 0$ for all $t \in [t,t+\rho]$. If $\xi \in L^\infty([t,t+\rho],\mathbb{R})$ satisfies

$$\xi(s) \leq \psi(s) + \int_t^s \phi(\tau)\xi(\tau)d\tau \quad \forall s \in [t,t+\rho],$$

then

$$\xi(s) \leq \psi(t)\exp\left(\int_t^s \phi(\tau)d\tau\right) + \int_t^s \psi'(\tau)\exp\left(\int_\tau^s \phi(\sigma)d\sigma\right)d\tau \quad \forall s \in [t,t+\rho].$$

A proof can be found in [12, Lemma 8.1], for example.

Lemma A.2.4 (Zorn) *Let $\mathcal{A} \neq \emptyset$ be a partially ordered set. If every totally ordered subset $\mathcal{O} \subset \mathcal{A}$ has an upper bound, then $\mathcal{A}$ has at least one maximal element.*

Lemma A.2.5 (Fatou's Lemma)
Let (f_n) be a sequence of non-negative measurable functions $I \longrightarrow \mathbb{R}$, then

$$\int_I \liminf_{n\to\infty} f_n(t)dt \leq \liminf_{n\to\infty} \int_I f_n(t)dt.$$

A proof can be found in [17, Lemma 2.18], for example. Observe that the form used in the proof of Theorem 3.2.1 is in fact the *Reverse* Fatou's Lemma which states that if, for the sequence (f_n), there is a non-negative measurable function $g\colon I \longrightarrow \mathbb{R}$ such that $f_n \leq g$ for all n and $\int_I g(t)dt < \infty$, then

$$\int_I \limsup_{n\to\infty} f_n(t)dt \geq \limsup_{n\to\infty} \int_I f_n(t)dt.$$

This result follows by applying Fatou's lemma to the sequence $(g - f_n)$.

Theorem A.2.6 (Lebesgue Dominated Convergence Theorem)
Let (f_n) be a sequence in $L^1(I,\mathbb{R}^m)$ such that the following two conditions hold:

(i) $f_n \longrightarrow f$ almost everywhere,

(ii) there exists a non-negative $g \in L^1(I,\mathbb{R})$ such that $\|f_n(t)\| \leq g(t)$ for almost all t and all n.

Then $f \in L^1(I,\mathbb{R}^m)$ and $\int_I f(t)dt = \lim_{n\to\infty}\int_I f_n(t)dt$.

A proof can be found in [17, Theorem 2.24], for example.

Let (X, Σ) be a measurable space and let μ be a measure on (X, Σ).

Definition A.2.7 *A signed measure on (X, Σ) is a function $\hat{\mu}\colon \Sigma \to [-\infty, \infty]$ such that*

(i) $\hat{\mu}(\emptyset) = 0$,

(ii) $\hat{\mu}$ *assumes at most one of the values* ∞ *and* $-\infty$,

(iii) if $\{E_j\}$ *is a sequence of disjoint sets in* Σ, *then*

$$\hat{\mu}\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \hat{\mu}(E_j),$$

where the latter sum converges absolutely if $\hat{\mu}(\cup_1^\infty E_j)$ *is finite.*

Theorem A.2.8 (Jordan Decomposition Theorem)

If $\hat{\mu}$ *is a signed measure, there exist unique measures* μ^+ *and* μ^- *such that* $\hat{\mu} = \mu^+ + \mu^-$ *and* $\mu^+ \perp \mu^-$.

For a proof, see [17, Theorem 3.4].

Definition A.2.9 *We define the total variation of a signed measure $\hat{\mu}$ to be the measure $|\hat{\mu}|$ given by*

$$|\hat{\mu}| = \mu^+ + \mu^-.$$

Appendix B

Technical results

B.1 Proof of Lemma 3.1.2

Proof.

STEP 1: Existence of a unique solution on a small interval.

By Property (iii) of $T \in \mathcal{T}_h^m$, there exist $\delta > 0$, $c_0 > 0$ and $\tau > t_0$ such that, for all $y, z \in \mathcal{C}(y^0; h, t_0, \tau, \delta)$,

$$\|(Ty)(t) - (Tz)(t)\| \leq c_0 \max_{s \in [t_0, \tau]} \|y(s) - z(s)\| \quad \text{for a.a. } t \in [t_0, \tau].$$

We may assume, without loss of generality, that $\delta \in (0,1)$ and $\tau - t_0 > 0$ are sufficiently small so that $[t_0, \tau] \times \overline{\mathbb{B}}_\delta(y^0(t_0)) \subset \mathcal{D}$. For each $\rho \in (t_0, \tau]$, define $C_\rho := \mathcal{C}(y^0, h, t_0, \rho, \delta)$ which, equipped with the metric

$$(y, z) \mapsto d_\rho(y, z) := \sup_{t \in [-h, \rho]} \|y(t) - z(t)\|,$$

is a complete metric space. Observe that, if $y \in C_\rho$, then $(t, y(t)) \in \mathcal{D}$ for all $t \in [t_0, \rho]$. For each $\rho \in (t_0, \tau]$, define the operator Z_ρ on C_ρ by

$$(Z_\rho y)(t) := \begin{cases} y^0(t), & t \in [-h, t_0], \\ y^0(t_0) + \int_{t_0}^t g(s, y(s), (Ty)(s)) ds, & t \in (t_0, \rho). \end{cases}$$

We proceed to show that Z_ρ is a contraction. Define $c_1 := \max_{s \in [-h, t_0]} \|y^0(s)\| + \delta$. By Property (iv) of $T \in \mathcal{T}_h^m$, there exists $c_2 > 0$ such that

$$\sup_{t \in [-h, \tau]} \|y(t)\| < c_1 \quad \Longrightarrow \quad \|(Ty)(t)\| < c_2 \quad \text{for a.a. } t \in [t_0, \tau].$$

By the local Lipschitz property of g, there exists a constant $c_3 > 0$ such that, for all $t \in [t_0, \tau]$,

$$\|g(t,y,w) - g(t,z,x)\| \le c_3\Big[\|y-z\| + \|w-x\|\Big] \quad \forall y,z \in \mathbb{B}_{c_1} \quad \forall w,x \in \mathbb{B}_{c_2}.$$

Write

$$g^* := \max\{\|g(t,y,w)\| \mid (t,y,w) \in [t_0,\tau] \times \overline{\mathbb{B}}_\delta(y^0(t_0)) \times \overline{\mathbb{B}}_{c_2}\}.$$

Fix $\rho^* \in (t_0, \tau]$ sufficiently close to t_0 so that

$$(\rho^* - t_0)(g^* + (c_0+1)c_3) < \delta.$$

Let $\rho \in (t_0, \rho^*]$ and $y \in C_\rho$. By definition, $(Z_\rho y)|_{[-h,t_0]} = y^0$ and

$$\begin{aligned}\|(Z_\rho y)(t) - y^0(t_0)\| &= \left\| \int_{t_0}^t g(s, y(s), (Ty)(s))ds \right\| \\ &\le \int_{t_0}^\rho \|g(s,y(s),(Ty)(s))\| ds \le (\rho - t_0)g^* < \delta \quad \forall t \in [t_0, \rho].\end{aligned}$$

Therefore $(Z_\rho y)(\cdot) \in C_\rho$. Furthermore,

$$\begin{aligned} d_\rho(Z_\rho y, Z_\rho z) &= \sup_{t\in[t_0,\rho]} \left\| \int_{t_0}^t [g(s,y(s),(Ty)(s)) - g(s,z(s),(Tz)(s))] ds \right\| \\ &\le \int_{t_0}^\rho \|g(s,y(s),(Ty)(s)) - g(s,z(s),(Tz)(s))\| ds \\ &\le (\rho - t_0)c_3 \left[\operatorname*{ess-sup}_{s\in[t_0,\rho]} \|(Ty)(s) - (Tz)(s)\| + d_\rho(y,z) \right] \\ &\le (c_0+1)(\rho - t_0)c_3\, d_\rho(y,z) \quad \forall\, y,z \in C_\rho. \end{aligned}$$

Since $(c_0+1)(\rho - t_0)c_3 < \delta < 1$, it follows that $Z_\rho \colon C_\rho \to C_\rho$ is a contraction. By the contraction mapping theorem, Z_ρ has a unique fixed point. We have shown that, for each $\rho \in (t_0, \rho^*]$, the initial-value problem (3.7) has a unique solution $y \in C_\rho$. We stress that the uniqueness property of y holds only in relation to solutions in the restricted class C_ρ: there may exist another solution on the interval $[-h, \rho]$ which is not contained in the space C_ρ. However, the following argument establishes uniqueness of the solution on a sufficiently small interval. Let y^* (not necessarily in C_{ρ^*}) be any solution on $[-h, \rho^*]$. Define

$$\Delta := \{t \in [t_0, \rho^*] \mid \|y^*(t) - y^0(t_0)\| = \delta\}, \quad \rho := \begin{cases} \inf \Delta, & \Delta \ne \emptyset, \\ \rho^*, & \Delta = \emptyset. \end{cases}$$

Clearly $\rho > t_0$ and $y := y^*|_{[-h,\rho]}$ is in C_ρ. Therefore, y is the unique solution of (3.7) on the interval $[-h, \rho]$.

STEP 2: Extended uniqueness: any two solutions must coincide on the intersection of their domains.

Let $y_1 : I_1 \to \mathbb{R}^m$ and $y_2 : I_2 \to \mathbb{R}^m$ be solutions of (3.7) and, without loss of generality, assume $I_2 \subset I_1$. For contradiction, suppose that $y_1|_{I_2} \neq y_2$. Let $t^* := \inf\{t \in I_2 \,|\, y_1(t) \neq y_2(t)\}$. By the result in Step 1, the solutions y_1 and y_2 must coincide on some interval $[-h, \rho]$, with $\rho > t_0$. Therefore, $t^* > t_0$. An application of the result of Step 1 in the context of an initial-value problem of the form (3.7), with t^* replacing t_0 and initial function $y_1|_{[-h,t^*]} \in C([-h, t^*], \mathbb{R}^m)$ replacing y^0, yields the existence of a unique solution $y \in C([-h, \rho], \mathbb{R}^m)$ for some $\rho > t^*$. It follows that $y_1(t) = y_2(t) = y(t)$ for all $t \in [-h, \rho]$, contradicting the definition of t^*.

STEP 3: Existence of a unique maximal solution.

Let $\mathcal{P}$ be the set of all $\rho > t_0$ such that there exists a solution y_ρ of (3.7) on the interval $[-h, \rho]$. By Step 1, we know that $\mathcal{P} \neq \emptyset$. Let $\omega := \sup \mathcal{P}$ and define $y\colon [-h, \omega) \to \mathbb{R}^m$ by the property

$$y|_{[-h,\rho]} = y_\rho \quad \forall \rho \in \mathcal{P}.$$

The function y is well-defined since, by Step 2, for all $\rho_1, \rho_2 \in \mathcal{P}$, we have $y_{\rho_2} = y_{\rho_1}|_{[-h,\rho_2]}$ whenever $\rho_2 \leq \rho_1$. Clearly y is a maximal solution and uniqueness follows by Step 2.

STEP 4: Assume that $y\colon [-h, \omega) \to \mathbb{R}^m$ is a maximal solution with $\omega < \infty$. Seeking a contradiction, suppose there exist $\sigma \in [t_0, \omega)$ and a compact set $\mathcal{K} \subset \mathcal{D}$ such that $(t, y(t)) \in \mathcal{K}$ for all $t \in [\sigma, \omega)$. Then y is bounded and, by Property (iv) of $T \in \mathcal{T}_h^m$, Ty is essentially bounded. Therefore, the function $t \mapsto (t, y(t), (Ty)(t))$ is essentially bounded and so, by continuity of g, it follows that $\dot{y}$ is essentially bounded on the interval $[t_0, \omega)$. Therefore y is uniformly continuous on $[-h, \omega)$ and so extends to $y^* \in C([-h, \omega], \mathbb{R}^m)$. By compactness of $\mathcal{K}$, we have $(\omega, y^*(\omega)) \in \mathcal{K} \subset \mathcal{D}$. An application of the result of Step 1 in the context of an initial-value problem of the form (3.7), with ω replacing t_0 and y^* replacing y^0, yields the existence of a unique solution $y^e \in C([-h, \rho], \mathbb{R}^m)$ for some $\rho > \omega$, with $y^e|_{[-h,\omega)} = y$. This contradicts maximality of y. $\square$

B.2 Proof of Theorem 7.4.2

Proof. Let $t_0 \in \mathbb{R}_+$ and $y^0 \in C([-h,t_0],\mathbb{R})$ be such that $(t,y^0(t)) \in \mathcal{D}$ for all $t \in [0,t_0]$.

STEP 1: First, we establish the existence of a unique solution on an interval $[-h,\rho]$ with $\rho > t_0$ sufficiently close to t_0. By Property (iii) of the operator class $\mathcal{T}_h^1$, there exist $\tau_0 > t_0$, $\delta_0 > 0$ and $c_0 > 0$ such that, for all $y_1, y_2 \in \mathcal{C}(y^0; h, t_0, \tau_0, \delta_0)$,

$$\text{ess-sup}_{t\in[t_0,\tau_0]} \|(Ty_1)(t) - (Ty_2)(t)\| \le c_0 \max_{t\in[t_0,\tau_0]} |y_1(t) - y_2(t)|.$$

We may assume that $\delta_0 \in (0,1)$ and $\tau_0 - t_0 > 0$ are sufficiently small so that

$$\mathcal{D}_0 := [t_0,\tau_0] \times [y^0(t_0) - \delta_0, y^0(t_0) + \delta_0] \subset \mathcal{D},$$

with φ bounded on the interval $[t_0,\tau_0]$. Next, consider the map

$$U\colon \mathcal{D} \to \mathbb{R}, \quad (t,z) \mapsto \nu(\alpha(\varphi(t)|z - r(t)|))\varphi(t)(z - r(t)).$$

Since α and ν are locally Lipschitz and by boundedness of r and φ on the interval $[t_0,\tau_0]$, it follows that there exists $c_1 > 0$ such that

$$|U(t,z_1) - U(t,z_2)| \le c_1|z_1 - z_2| \quad \forall (t,z_1) < (t,z_2) \in \mathcal{D}_0.$$

For each $\rho \in (t_0,\tau_0]$, define $\mathcal{C}_\rho^0 := \mathcal{C}(y^0; h, t_0, \rho, \delta_0)$. Observe that, if $y \in \mathcal{C}_\rho^0$, then $(t,y(t)) \in \mathcal{D}_0$ for all t such that $t_0 \le t \le \rho \le \tau_0$. Therefore, for each $\rho \in [t_0,\tau_0]$, we may define an operator $\mathbf{U}_\rho\colon \mathcal{C}_\rho^0 \to C([0,\rho],\mathbb{R})$ by

$$(\mathbf{U}_\rho y)(t) := U(t,y(t)) \quad \forall t \in [0,\rho]$$

and record the following fact:

$$|(\mathbf{U}_\rho y_1)(t) - (\mathbf{U}_\rho y_2)(t)| \le c_1|y_1(t) - y_2(t)| \quad \forall t \in [0,\rho] \quad \forall y_1, y_2 \in \mathcal{C}_\rho^0. \tag{B.1}$$

Define $w \in C([0,t_0],\mathbb{R})$ by

$$w(t) := U(t,y^0(t)) \quad \forall t \in [0,t_0].$$

In particular,

$$(\mathbf{U}_\rho y)(t) = w(t) \quad \forall t \in [0,t_0] \quad \forall y \in \mathcal{C}_\rho^0.$$

By hypothesis (i) of the hysteresis operator class $\mathcal{O}$, there exist $\tau_1 \in (t_0, \tau_0]$, $\delta_1 \in (0, \delta_0]$ and $c_2 > 0$ such that

$$\max_{t\in[0,\tau_1]} |(\Phi v_1)(t) - (\Phi v_2)(t)| \le c_2 \max_{t\in[0,\tau_1]} |v_1(t) - v_2(t)| \quad \forall v_1, v_2 \in \mathcal{C}(w; 0, t_0, \tau_1, \delta_1). \quad \text{(B.2)}$$

Furthermore, by continuity of U, there exist $\tau_2 \in (t_0, \tau_1]$ and $\delta_2 \in (0, \delta_0]$ such that if $\rho \in (t_0, \tau_2]$, then

$$(\mathbf{U}_\rho y)(t) \in \mathcal{C}(w; 0, t_0, \rho, \delta_1) \quad \forall y \in \mathcal{C}(y^0; h, t_0, \rho, \delta_2) \subset \mathcal{C}^0_\rho. \quad \text{(B.3)}$$

For each $\rho \in (t_0, \tau_2]$, we define $\mathcal{C}_\rho := \mathcal{C}(y^0; h, t_0, \rho, \delta_2)$. Combining (B.3) with (B.1) and (B.2), we may conclude that there exists $c_3 > 0$ such that, for every $\rho \in (t_0, \tau_2]$,

$$\max_{t\in[0,\rho]} |(\Phi(\mathbf{U}_\rho y_1))(t) - (\Phi(\mathbf{U}_\rho y_2))(t)| \le c_3 \max_{t\in[0,\rho]} |y_1(t) - y_2(t)| \quad \forall y_1, y_2 \in \mathcal{C}_\rho. \quad \text{(B.4)}$$

Furthermore, as a consequence of (B.4), there exists $c_4 > 0$ such that, for every $\rho \in (t_0, \tau_2]$

$$|(\Phi(\mathbf{U}_\rho y))(t)| \le c_4 \quad \forall t \in [0, \rho] \quad \forall y \in \mathcal{C}_\rho.$$

Equipped with the metric

$$(y_1, y_2) \mapsto d_\rho(y_1, y_2) := \max_{t\in[-h,\rho]} |y_1(t) - y_2(t)|,$$

the metric space $\mathcal{C}_\rho$ is complete. Now, for each $\rho \in (t_0, \tau_2]$, define the operator $\mathbf{C}_\rho$ on $\mathcal{C}_\rho$ by

$$(\mathbf{C}_\rho y)(t) := \begin{cases} y^0(t), & t \in [-h, t_0], \\ y^0(t_0) + \int_{t_0}^t f(p(s), (Ty)(s)) + \beta(\Phi(\mathbf{U}_\rho y))(s)ds, & t \in (t_0, \rho). \end{cases}$$

We proceed by showing that there exists $\rho^* \in (t_0, \tau_2]$ such that, for all $\rho \in (t_0, \rho^*]$, $\mathbf{C}_\rho(\mathcal{C}_\rho) \subset \mathcal{C}_\rho$ and $\mathbf{C}_\rho$ is a contraction (and so, for each such ρ, $\mathbf{C}_\rho$ has a unique fixed point). By Property (iv) of $T \in \mathcal{T}^1_h$, there exists $c_5 > 0$ such that, for every $\rho \in (t_0, \rho^*]$,

$$\|(Ty)(t)\| < c_5 \quad \text{for a.a. } t \in [t_0, \rho] \quad \forall y \in \mathcal{C}_\rho.$$

By the local Lipschitz property of f and essential boundedness of p, there exists a constant $c_6 > 0$ such that

$$|f(p(t), x_1) - f(p(t), x_2)| \le c_6 \|x_1 - x_2\| \quad \text{for a.a. } t \in [t_0, \tau_2] \text{ and all } x_1, x_2 \in \mathbb{R}^q \text{ with } \|x_1\|, \|x_2\| \le c_5.$$

Write

$$c_7 := \max\{|f(q, x)| \mid \|q\| \le \|p\|_\infty, \|x\| \le c_5\}.$$

Fix $\rho^* \in (t_0, \tau_2]$ sufficiently close to t_0 so that

$$(\rho^* - t_0)(c_7 + c_0 c_6 + c_3|\beta| + c_4|\beta|) \le \delta_2.$$

Let $\rho \in (t_0, \rho^*]$ and $y \in \mathcal{C}_\rho$. By definition, $(\mathbf{C}_\rho y)|_{[-h,t_0]} = y^0$ and

$$\begin{aligned}
|(\mathbf{C}_\rho y)(t) - y^0(t_0)| &= \left| \int_{t_0}^{t} f(p(s), (Ty)(s)) + \beta\,(\Phi(\mathbf{U}_\rho y))(s) ds \right| \\
&\le \int_{t_0}^{\rho} |f(p(s), (Ty)(s)) + \beta\,(\Phi(\mathbf{U}_\rho y))(s)| ds \\
&\le (\rho - t_0)(c_7 + c_4|\beta|) \le \delta_2 \quad \forall t \in [t_0, \rho].
\end{aligned}$$

Therefore $(\mathbf{C}_\rho y)(\cdot) \in \mathcal{C}_\rho$, establishing the fact that $\mathbf{C}_\rho(\mathcal{C}_\rho) \subset \mathcal{C}_\rho$ for all $\rho \in (t_0, \rho^*]$. Furthermore, for $\rho \in (t_0, \rho^*]$ and $y_1, y_2 \in \mathcal{C}_\rho$,

$$\begin{aligned}
d_\rho(\mathbf{C}_\rho y_1, \mathbf{C}_\rho y_2) &= \sup_{t \in [t_0, \rho]} \left| \int_{t_0}^{t} \big[f(p(s), (Ty_1)(s)) - f(p(s), (Ty_2)(s)) \right. \\
&\qquad \left. + \beta(\Phi(\mathbf{U}_\rho y_1))(s) - \beta(\Phi(\mathbf{U}_\rho y_2))(s) \big] ds \right| \\
&\le \int_{t_0}^{\rho} |f(p(s), (Ty_1)(s)) - f(p(s), (Ty_2)(s)) \\
&\qquad + \beta(\Phi(\mathbf{U}_\rho y_1))(s) - \beta(\Phi(\mathbf{U}_\rho y_2))(s)| ds \\
&\le (\rho - t_0) \Big[\sup_{t \in [t_0, \rho]} |f(p(t), (Ty_1)(t)) - f(p(t), (Ty_2)(t))| \\
&\qquad + |\beta| \sup_{t \in [t_0, \rho]} |(\Phi(\mathbf{U}_\rho y_1))(t) - (\Phi(\mathbf{U}_\rho y_2))(t)| \Big] \\
&\le (\rho - t_0)(c_0 c_6 + c_3|\beta|) d_\rho(y_1, y_2).
\end{aligned}$$

Since $(\rho - t_0)(c_0c_6 + c_3|\beta|) \le \delta_2 < 1$, it follows that $\mathbf{C}_\rho\colon \mathcal{C}_\rho \to \mathcal{C}_\rho$ is a contraction. Therefore, by the contraction mapping theorem, $\mathbf{C}_\rho$ has a unique fixed point. We have shown that, for each $\rho \in (t_0, \rho^*]$, the initial-value problem (7.11) has a unique solution in $\mathcal{C}_\rho$. We stress that the uniqueness property of y holds only in relation to solutions in the restricted class $\mathcal{C}_\rho$: there may exist other solutions on the interval $[-h, \rho]$ which are not contained in the space $\mathcal{C}_\rho$. However, the following argument establishes uniqueness of the solution on a sufficiently small interval. Let y^* (not necessarily in $\mathcal{C}_{\rho^*}$) be any solution on $[-h, \rho^*]$. Define

$$\Delta := \{t \in [t_0, \rho^*] \,|\, |y^*(t) - y^0(t_0)| = \delta\}, \quad \rho := \begin{cases} \inf \Delta, & \Delta \neq \emptyset, \\ \rho^*, & \Delta = \emptyset. \end{cases}$$

Clearly $\rho > t_0$ and $y := y^*|_{[-h,\rho]}$ is in $\mathcal{C}_\rho$. Therefore, y is the unique solution of (7.11) on the interval $[-h, \rho]$.

STEP 2: Extended uniqueness: any two solutions must coincide on the intersection of their domains.

Let $y_1\colon I_1 \to \mathbb{R}$ and $y_2\colon I_2 \to \mathbb{R}$ be solutions of (7.11) and, without loss of generality, assume $I_2 \subset I_1$. For contradiction, suppose that $y_1|_{I_2} \neq y_2$. Let $t^* := \inf\{t \in I_2 \,|\, y_1(t) \neq y_2(t)\}$. By the result in Step 1, the solutions y_1 and y_2 must coincide on some interval $[-h, \rho]$, with $\rho > t_0$. Therefore, $t^* > t_0$. An application of the result of Step 1 in the context of an initial-value problem of the form (7.11), with t^* replacing t_0 and initial function $y_1|_{[-h,t^*]} \in C([-h, t^*], \mathbb{R})$ replacing y^0, yields the existence of a unique solution $y \in C([-h, \rho], \mathbb{R})$ for some $\rho > t^*$. It follows that $y_1(t) = y_2(t) = y(t)$ for all $t \in [-h, \rho]$, contradicting the definition of t^*.

STEP 3: Existence of a unique maximal solution.

Let $\mathcal{P}$ be the set of all $\rho > t_0$ such that there exists a solution y_ρ of (7.11) on the interval $[-h, \rho]$. By Step 1, we know that $\mathcal{P} \neq \emptyset$. Let $\omega := \sup \mathcal{P}$ and define $y\colon [-h, \omega) \to \mathbb{R}$ by the property

$$y|_{[-h,\rho]} = y_\rho \quad \forall \rho \in \mathcal{P}.$$

The function y is well-defined since, by Step 2, for all $\rho_1, \rho_2 \in \mathcal{P}$, we have $y_{\rho_2} = y_{\rho_1}|_{[-h,\rho_2]}$ whenever $\rho_2 \le \rho_1$. Clearly y is a maximal solution and uniqueness follows by Step 2.

STEP 4: Assume that $y\colon [-h, \omega) \to \mathbb{R}$ is a maximal solution with $\omega < \infty$. Seeking a contradiction, suppose $\limsup_{t \to \omega} \varphi(t)|y(t) - r(t)| < 1$. Therefore, k, and hence u, are bounded. By Property (iv) of the operator class $\mathcal{T}_h^1$, Ty is essentially bounded and, by Property (ii) of the hysteresis class $\mathcal{O}$, Φu is bounded. From the differential equation in (7.11), it now follows that $\dot{y}$ is essentially bounded on $[0, \omega)$. Therefore, y is uniformly continuous on $[-h, \omega)$ and so extends to $y^* \in C([-h, \omega], \mathbb{R})$. Furthermore,

$$\varphi(\omega)|y^*(\omega) - r(\omega)| = \lim_{t \to \omega} \varphi(t)|y^*(t) - r(t)| = \limsup_{t \to \omega} \varphi(t)|y(t) - r(t)| < 1,$$

showing that $(\omega, y^*(\omega)) \in \mathcal{D}$. An application of the result in Step 1 in the context of an initial-value problem of the form (7.11), with ω replacing t_0 and y^* replacing y^0, yields the existence of a unique solution $y^e \in C([-h, \rho], \mathbb{R})$ for some $\rho > \omega$, with $y^e|_{[-h,\omega)} = y$. This contradicts maximality of y. $\square$

Appendix C

Set-valued analysis

C.1 Set-valued maps and upper semicontinuity

In what follows, X and Y are non-empty subsets of finite-dimensional Euclidean spaces. We first introduce the concept of a set-valued map.

Definition C.1.1 (Set-valued map)
A set-valued map F between X and the subsets of Y is a map that assigns a non-empty subset $F(x) \subset Y$ to each element $x \in X$ (the values of F are the sets $F(x)$ for $x \in X$).

A set-valued map is said to have convex values if $F(x)$ is convex for all $x \in X$ and compact values if $F(x)$ is compact for all $x \in X$.

Definition C.1.2 (Graph of a set-valued map)
The graph of a set-valued map F is defined as

$$\mathit{graph}(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}.$$

Definition C.1.3 (Upper semicontinuity)
A set-valued map F is said to be upper semicontinuous at $x^0 \in X$ if, for any open neighbourhood N of $F(x^0)$, there exists an open neighbourhood M of x^0 with $F(M) \subset N$. F is said to be upper semicontinuous if it is so at every $x^0 \in X$.

A second concept, upper semicontinuity in the 'ε-sense', is defined as follows.

Definition C.1.4 *A set-valued map F is said to be upper semicontinuous in the ε-sense if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $F(x^0 + \mathbb{B}_\delta) \subset F(x^0) + \mathbb{B}_\varepsilon$.*

Remark C.1.5 *In the case when F has compact values, the two above definitions of upper semicontinuity coincide.*

We now assemble three important results used in the proof of Theorem 3.2.1 in Chapter 3.

Proposition C.1.6 *Let F be an upper semicontinuous, set-valued map on X with closed values in Y, then $graph(F)$ is closed.*

A proof can be found in [2, Proposition 2, Page 41].

Proposition C.1.7 *Let F be an upper semicontinuous, set-valued map on X, with compact values in Y. If $\mathcal{K} \subset X$ is compact, then $F(\mathcal{K}) := \cup_{z\in\mathcal{K}} F(z)$ is compact.*

For a proof, see [2, Proposition 3, Page 42].

The third result is the Approximate Selection Theorem (see for example [2, Theorem 1, Page 84]). Given a set-valued map F on X with values in Y, a selection for F is a function $x \mapsto f(x)$ with the property that $f(x) \in F(x)$ for all $x \in X$. An upper semicontinuous map may not always possess a continuous selection, for instance, the following (well-known) counter-example:

$$F(x) := \begin{cases} \{-1\}, & \text{for } x < 0, \\ [-1, 1], & \text{for } x = 0, \\ \{1\}, & \text{for } x > 0, \end{cases}$$

admits no continuous selection. Instead, we make use of the Approximate selection theorem.

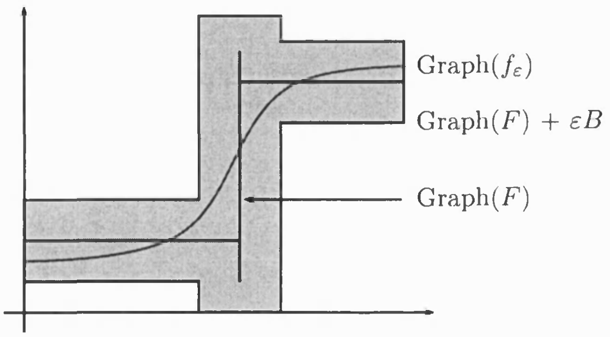


Figure 1: Approximate selection for a set-valued map.

Theorem C.1.8 (Approximate Selection Theorem)
Let F be an upper semicontinuous, set-valued map from X to the convex subsets of

Y. Then, for every $\varepsilon > 0$, there exists a locally Lipschitz map $f_\varepsilon \colon X \to Y$ with range contained in the convex hull of the range of F, and:

$$Graph(f_\varepsilon) \subset Graph(F) + \varepsilon B.$$

The proof can be found in [2, Theorem 1, Page 84], for example.

C.2 Support functions

Let $C \subset \mathbb{R}^m$ be a non-empty, closed, convex set.

Definition C.2.1 *The function $\sigma_C \colon \mathbb{R}^m \to \mathbb{R}$ given by*

$$\sigma_C(q) := \sup\{\langle q, \zeta\rangle \mid \zeta \in C\}$$

is the support function of C.

Theorem C.2.2 *Let $\sigma_C \colon \mathbb{R}^m \to \mathbb{R}$ be the support function of the non-empty, closed, convex set C, then the following hold.*

(i) $\zeta \in C \iff \langle q, \zeta\rangle \le \sigma_C(q)$ for all $q \in \mathbb{R}^m$,

(ii) if C is compact, then σ_C is globally Lipschitz.

For a proof, see [39, Corollary 2D.2].

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